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GENERAL SOLUTION TO CONSTANT GAIN TRACKING FILTERS

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This report presents a general solution to Constant Gain Tracking Filters, which are nothing more than matrix difference equations. These equations can be solved regardless of the measurement model provided the model is deterministic. The general solution can then be used to compute the covariance matrix under very general assumptions about the noise. In this report, the general case will be solved, and the α–β filter will be solved as an illustrative example.

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FOREWORD

The Naval Surface Warfare Center Dahlgren Division (NSWCDD) has a growing responsibility for providing ways to accurately predict and detect targets. Information processed through track filters is the heart of all major combat systems. Tracking and estimation technologies are essential to the future of the United States Navy. Currently, combat system functions such as, Command and Decision, Air Control and Combat Systems Component Individual Error Checking, rely heavily on track processing data. In the future, combat system functions such as, Common Command and Decision, Multi-Platform Weapons Control, Common Air Picture and Mission Expansion, will rely on track processing data.

This report has been reviewed at NSWCDD by A. Riedl, Head, Combat Systems Technology Division (B30); and M. Kuchinski, Head, Digital Systems Branch (B32).

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1 INTRODUCTION

The usage of tracking filters associated with a tracking radar dates back to work by Sklansky [20]. He proposed measures of performance including stability, transient response, noise and maneuver error as a function of the dynamic parameters α and β. All of the work was based on a frequency domain or z-transform analysis. Subsequent work by Benedict-Bordner [2] proposed a relationship between α and β based on a pole-matching technique that combined transient performance and noise reduction capability. Subsequent analysis was performed by Simpson [19], Neal, and Benedict [18] for the $\alpha-\beta-\gamma$ filter. By this time, the Kalman filter was becoming well known in the radar community. Thereafter, the tendency was to discuss the $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters as steady state solution to the Kalman filter. Subsequent papers by Friedland [17], Fitzgerald [6], and Kalata [13] exploited this formal similarity to derive many results that can be used to characterize tracking performance in a multi-tracking environment. The basis for the analysis of performance used internally with the Aegis community is summarized in the internal manual entitled "The Working Engineers Guide To $\alpha - \beta$ and $\alpha - \beta - \gamma$ Filters" by Reifler and Solomon [23]. Later, much of this work was summarized in the open literature by Kalata [22]. A summary of subsequent developments in the literature to 1992 is found in Kalata [22] with some additional work since then found in Gray ([10], [11]), as well as the open literature.

Contrary to the approach that is usually taken in the literature, we propose that the more 'natural' viewpoint is to introduce the constant gain filter and an entity that is independent of the Kalman filter. One can derive the information that characterized the filter performance without regard to the Kalman filter design criteria. One can then show how the performance criteria generalize naturally to the Kalman filter or to a variation of the Kalman filter that replaces the process noise with a bias reduction criteria. While variations on the filters discussed are currently in use within naval systems, it is likely that they will be replaced with much more advanced estimation techniques such as the interacting multiple model (IMM) filter. Exploring the ability to bound filter performance is a necessary part of the redesign to replace existing filters with advanced filter architectures. An IMM design that consists of several $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters can be used to provide such a bound without the requirements of a detailed system simulation [21]. Different architectures can be explored by this approach, so that tight error bounds can be determined as part of an overall system performance. Subsequent implementation of a true IMM would then be used and known to have performance boundaries within this boundary. We will take the results derived here and explore such issues in subsequent reports.

The $\alpha - \beta$ filter has found application when large numbers of objects are to be tracked. By clever selection of the gains, and careful design, variable gain $\alpha - \beta$ filters combine sufficient elements of the Kalman filter ([8],[1]) so that there is not significant tracking degradation. Thus, there is useful information to be gained by a detailed performance characterization of the filter.

The tracking equations for the $\alpha - \beta$ filter consist of two parts: prediction equations, which are given by

$$x_p(k) = x_s(k-1) + v_s(k-1)T \tag{1-1}$$

$$v_p(k) = v_s(k-1) \tag{1-2}$$

and smoothing equations, which are given by

$$x_s(k) = x_p(k) + \alpha(x_m(k) - x_p(k)) \tag{1-3}$$

$$v_s(k) = v_P(k) + \frac{\beta}{T}(x_m(k) - x_p(k)) \tag{1-4}$$

where

- $x_s(k)$ = smoothed position at the k-th interval
- $x_p(k)$ = predicted position at the k-th interval
- $x_m(k)$ = measured position at the k-th interval
- $v_s(k)$ = smoothed velocity at the k-th interval
- $v_p(k)$ = predicted velocity at the k-th interval
- T = radar update interval or period
- α, β = filter weighing coefficients

Alternatively, these equations can be written as

$$|x_s\rangle_k = F_\beta \, |x_s\rangle_{k-1} + G_\beta x_m(k) \tag{1-5}$$

where

$$F_\beta = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\frac{\beta}{T} & 1-\beta \end{bmatrix} \tag{1-6}$$

$$|x_s\rangle_k = \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} \tag{1-7}$$

$$G_\beta = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \end{bmatrix} \tag{1-8}$$

These filter equations are one-dimensional, but can be extended to three dimensions by substituting successively y and z for x in Eq. (1-1) through Eq. (1-4). The filter equations are usually analyzed in one dimension and the resulting analysis is usually extended to three dimensions with the assumption that similar results are given.

For the class of problems when this occurs, the filter can be viewed as a constant gain filter which is nothing more than a matrix difference equation. This equation can then be solved regardless of the measurement model provided the model is deterministic. The general solution can then be used to compute the covariance matrix under very general assumptions about the noise. This is an alternative and slightly more general to work done by Fitzgerald [6],[7], in the early eighties. In this report, the general case will be solved first, and the $\alpha - \beta$ filter [2] will be solved as an illustrative example. Previously [9], the z-transform or frequency domain method was used, but here the direct methods that have become more fashionable in recent years will be used. The solutions are independent of the particular relationship between α and β that are discussed in [2] and [13].

In general, the update equations for a constant gain filter can be written as

$$\eta_{n+1} = F\eta_n + x_{n+1}G, \tag{1-9}$$

where

$$\eta_n \doteq n^{th} \textit{ vector state measurement},$$

$$F, G \doteq \textit{filter update, gain matrices},$$

$$x_n \doteq \textit{scalar measurement model}.$$

One would like to solve Eq. (1-9) under very general circumstances, which will be demonstrated in the next section. This method would apply to any scalar measurement model.

2 SOLUTIONS TO GENERALIZED FILTER EQUATIONS

In general, the update equations for a constant gain filter can be written as

$$\eta_{n+1} = F\eta_n + x_{n+1}G, \tag{2-1}$$

where,

$$\eta_n \doteq n^{th} \; vector \; state \; measurement$$
$$F, G \doteq filter \; update, gain \; matrices,$$
$$x_n \doteq scalar \; measurement \; model.$$

Hence, for $n = 0$, Eq. (2-1) has the form

$$\eta_1 = F\eta_0 + x_1 G \tag{2-2}$$

For $n = 1$, Eq. (2-1) has the form

$$\eta_2 = F\eta_1 + x_2 G \tag{2-3}$$

Substituting the value of η_1, Eq. (2-2), into Eq. (2-3) gives

$$\begin{aligned} \eta_2 &= F(F\eta_0 + x_1 G) + x_2 G \\ &= F^2\eta_0 + F x_1 G + x_2 G \end{aligned} \tag{2-4}$$

For $n = 2$,

$$\eta_3 = F\eta_2 + x_3 G \tag{2-5}$$

Substituting the value of η_2, Eq. (2-4), into Eq. (2-5) gives

$$\begin{aligned} \eta_3 &= F(F^2\eta_0 + F x_1 G + x_2 G) + x_3 G \\ &= F^3\eta_0 + F^2 x_1 G + F x_2 G + x_3 G \end{aligned} \tag{2-6}$$

and so on. Thus one has a basis for induction, and can then establish the following theorem:

Theorem: The general solution to Eq. (2-1) is

$$\eta_k = F^k\eta_0 + \sum_{n=1}^{k} F^{k-n} G x_n. \tag{2-7}$$

Proof: Use induction or substitution to verify the solution directly, as done above.

One can recognize that in the theorem the general solution is the combination of the homogeneous and inhomogeneous solutions. The homogeneous solution, found for x_n equal to zero, is

$$\eta_k^h = F^k\eta_0. \tag{2-8}$$

The inhomogeneous solution is

$$\eta_k^i = \sum_{n=1}^{k} F^{k-n} G x_n \tag{2-9}$$

To proceed further, the powers of the matrices need to be explicitly evaluated.

Let

$$f(\lambda) = \det[F - \lambda I], \tag{2-10}$$

by the Caley-Hamilton theorem,

$$f(F) = 0 = \gamma_0 I + \gamma_1 F + ... + \gamma_m F^m \tag{2-11}$$

where m is the order of the matrix. Thus, any power $k > m$ of a matrix can be written as

$$F^k = \gamma_0(k) I + \gamma_1(k) F + ... + \gamma_m(k) F^m = \langle F| \, |\gamma(k)\rangle \tag{2-12}$$

where

$$|F\rangle = \begin{bmatrix} I \\ F \\ . \\ , \\ F^m \end{bmatrix} \; and \; |\gamma(k)\rangle = \begin{bmatrix} \gamma_0(k) \\ \gamma_1(k) \\ . \\ . \\ \gamma_m(k) \end{bmatrix} \tag{2-13}$$

(note $\langle A| = |A\rangle^t$ where t denotes transpose.) Note the solution in Eq. (2-11) can be simplified by expressing F^k as in Eq.(2-12) provided one can evaluate sums of the form $\sum_n \gamma(k) x_n$. Therefore the solution is of the form

$$\eta_k = \langle F \mid \gamma(k)\rangle \, \eta_0 + \sum_{n=1}^{k} \langle F \mid \gamma(k-n)\rangle \, G x_n. \tag{2-14}$$

3 $\alpha - \beta$ Filter

The $\alpha - \beta$ filter has found application when large numbers of objects are to be tracked. Thus, there is useful information to be gained by a detailed performance characterization of the filter. The tracking equations for the $\alpha - \beta$ filter consists of two parts: prediction equations, which are given by

$$x_p(k) = x_s(k-1) + v_s(k-1)T \tag{3-1}$$

$$v_p(k) = v_s(k-1) \tag{3-2}$$

and smoothing equations, which are given by

$$x_s(k) = x_p(k) + \alpha(x_m(k) - x_p(k)) \tag{3-3}$$

$$v_s(k) = v_p(k) + \frac{\beta}{T}(x_m(k) - x_p(k)) \tag{3-4}$$

where

- $x_s(k)$ = smoothed position at the k-th interval
- $x_p(k)$ = predicted position at the k-th interval
- $x_m(k)$ = measured position at the k-th interval
- $v_s(k)$ = smoothed velocity at the k-th interval
- $v_p(k)$ = predicted velocity at the k-th interval
- T = radar update interval or period
- α, β = filter weighing coefficients

The filter gains, α and β satisfy the following relation

$$0 < \beta \leq \alpha < 1 \tag{3-5}$$

There are three commonly used relationships between α and β. The first is the Kalata relation, which is obtained from steady state Kalman filter theory assuming zero mean white noise in the position and velocity state equations [13].

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{3-6}$$

The second is the Benedict-Bordner relation, which is derived based on good noise reduction and good tracking through maneuvers.

$$\beta_{BB} = \frac{\alpha^2}{2-\alpha} \tag{3-7}$$

The third is the Continuous White Noise (CTWN) relation.

$$\alpha = \sqrt{2\beta + \frac{\beta^2}{12}} - \frac{\beta}{2} \tag{3-8}$$

Alternatively, Eq. (3-1) through Eq. (3-4) can be written as

$$|x_s\rangle_k = H_\beta\,|x_p\rangle_k + G_\beta x_m(k) \tag{3-9}$$

and

$$|x_p\rangle_{k+1} = Q_\beta\,|x_s\rangle_k \tag{3-10}$$

where

$$|x_s\rangle_k = \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} \tag{3-11}$$

$$|x_p\rangle_k = \begin{bmatrix} x_p(k) \\ v_p(k) \end{bmatrix} \tag{3-12}$$

$$Q_\beta = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{3-13}$$

$$H_\beta = \begin{bmatrix} 1-\alpha & 0 \\ -\frac{\beta}{T} & 1 \end{bmatrix} \tag{3-14}$$

$$G_\beta = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \end{bmatrix} \tag{3-15}$$

Alternatively, Eq. (3-9) can be written as

$$|x_s\rangle_k = F_\beta\,|x_s\rangle_{k-1} + G_\beta x_m(k) \tag{3-16}$$

where

$$F_\beta = H_\beta \cdot Q_\beta = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\frac{\beta}{T} & 1-\beta \end{bmatrix} \tag{3-17}$$

The eigenvalues of F_β can be shown to satisfy the equation

$$f(\lambda) = 0 = (1-\alpha-\lambda)(1-\beta-\lambda) + \beta(1-\alpha), \tag{3-18}$$

which simplifies to

$$f(\lambda) = 0 = \lambda^2 + (\alpha+\beta-2)\lambda + (1-\alpha) \tag{3-19}$$

Let $r = \sqrt{1-\alpha}$ and $2r\cos(\theta) = 2-\alpha-\beta$, then

$$f(\lambda) = 0 = \lambda^2 - 2r\cos(\theta)\lambda + r^2, \tag{3-20}$$

or

$$f(\lambda) = 0 = (\lambda - re^{j\theta})(\lambda - re^{-j\theta}). \tag{3-21}$$

Since F_β is a two by two matrix, it can be written as

$$F^k = \gamma_0(k) + \gamma_1(k)F_\beta. \tag{3-22}$$

The eigenvalues are

$$\lambda_\pm = re^{\pm j\theta}. \tag{3-23}$$

Using the method in [5] for determining the power of a matrix gives two simultaneous equations with two unknowns. Define

$$g(\lambda) = \gamma_0(k) + \gamma_1(k)\lambda \tag{3-24}$$

and

$$h(\lambda) = \lambda^k. \tag{3-25}$$

When $h(\lambda_0) = g(\lambda_0)$, one gets

$$r^k e^{jk\theta} = \gamma_0(k) + \gamma_1(k)re^{j\theta}, \tag{3-26}$$

and when $h(\lambda_1) = g(\lambda_1)$, one gets

$$r^k e^{-jk\theta} = \gamma_0(k) + \gamma_1(k)re^{-j\theta}. \tag{3-27}$$

Therefore, there exists two simultaneous equations to solve for the two eigenvalues to get the coefficients. Solving these two equations give the result

$$\gamma_0(k) = \frac{-r^k \sin((k-1)\theta}{\sin(\theta)}, \tag{3-28}$$

and

$$\gamma_1(k) = \frac{r^{k-1}\sin(k\theta)}{\sin(\theta)}. \tag{3-29}$$

The general solution to the $\alpha - \beta$ filter equations consists of the homogeneous solutions and inhomogeneous equations. As stated previously, the homogeneous solution is of the form

$$\eta_k^h = F^k \eta_0, \tag{3-30}$$

where

$$F^k = \gamma_0(k)I + \gamma_1(k)F_\beta, \tag{3-31}$$

therefore

$$\eta_k^h = [\gamma_0(k)I + \gamma_1(k)F_\beta]\,\eta_0. \tag{3-32}$$

By substituting the known values for $\gamma_0(k)$ and $\gamma_1(k)$, the homogeneous solution takes the form

$$\eta_k^h = \left[\frac{-r^k \sin((k-1)\theta)I}{\sin(\theta)} + \frac{r^{k-1}\sin(k\theta)F_\beta}{\sin(\theta)}\right]\eta_0. \tag{3-33}$$

Also, inhomogeneous solutions are of the form

$$\eta_k^i = \sum_{n=1}^{k} F^{k-n} x_n G_\beta, \tag{3-34}$$

where

$$F^{k-n} = \gamma_0(k-n)I + \gamma_1(k-n)F_\beta \tag{3-35}$$

therefore

$$\begin{aligned} \eta_k^i &= \sum_{n=1}^{k} [\gamma_0(k-n)I + \gamma_1(k-n)F_\beta] x_n G_\beta, \\ &= \sum_{m=0}^{k-1} [\gamma_0(m)I + \gamma_1(m)F_\beta] x_{k-m} G_\beta. \end{aligned} \tag{3-36}$$

Again, by substituting the known values for $\gamma_0(k)$ and $\gamma_1(k)$, the inhomogeneous solution takes the form

$$\eta_k^i = \frac{1}{\sin\theta} \sum_{m=0}^{k-1} [-r^m \sin((m-1)\theta)I + r^{m-1}\sin(m\theta)F_\beta] x_{k-m} G_\beta. \tag{3-37}$$

One can further simplify the inhomogeneous solutions in Eq. (3-37) by using the trigonometric identity

$$\sin[(m-1)\theta] = \sin(m\theta)\cos(\theta) - \cos(m\theta)\sin(\theta), \tag{3-38}$$

therefore

$$\eta_k^i = \sum_{m=0}^{k-1} \left[\left(\frac{\left(-\cos(\theta)\cdot I + \frac{F_\beta}{r}\right)}{\sin(\theta)} r^m \sin(m\theta) + (r^m \cos(m\theta))\, I \right) x_{k-m} \right] G_\beta. \tag{3-39}$$

Therefore the general solution, which is the summation of homogeneous and inhomogeneous solutions gives

$$\begin{aligned} n_k &= \frac{1}{\sin(\theta)} \left[\left(-r^k \sin((k-1)\theta)\right) I + \left(r^{k-1}\sin(k\theta)\right) F_\beta \right] \eta_0 \\ &\quad + \sum_{m=0}^{k-1} \left(\left[\frac{A}{\sin(\theta)} r^m \sin(m\theta) + (r^m\cos(m\theta))\, I \right] x_{k-m} \right) G_\beta, \end{aligned} \tag{3-40}$$

where

$$A = -\cos(\theta)I + \frac{F_\beta}{r}. \tag{3-41}$$

3.1 CONSTANT MOTION MODEL

If one assumes that x_{k-m} is a constant, say x_0, then the general solution for the Constant Motion Model becomes

$$\begin{aligned} n_k &= \frac{1}{\sin(\theta)}\left[\left(-r^k \sin((k-1)\theta)\right) I + \left(r^{k-1}\sin(k\theta)\right) F_\beta\right]\eta_0 \\ &+ \sum_{m=0}^{k-1}\left(\frac{A}{\sin(\theta)}r^m \sin(m\theta) + (r^m \cos(m\theta))\, I\right) x_0 G_\beta. \end{aligned} \tag{3-42}$$

Examine only the inhomogeneous solution

$$\begin{aligned} \eta_k^i &= \sum_{m=0}^{k-1}\left(\frac{A}{\sin(\theta)}r^m \sin(m\theta) + (r^m \cos(m\theta))\, I\right) x_0 G_\beta \\ &= \left[\frac{A}{\sin(\theta)}\sum_{m=0}^{k-1} r^m \sin(m\theta) + \sum_{m=0}^{k-1} r^m \cos(m\theta) I\right] x_0 G_\beta \\ &= \left[\frac{A}{\sin(\theta)} S(r,\theta) + C(r,\theta) I\right] x_0 G_\beta \end{aligned} \tag{3-43}$$

where

$$S(r,\theta) = \sum_{m=0}^{k-1} r^m \sin(m\theta) \tag{3-44}$$

and

$$C(r,\theta) = \sum_{m=0}^{k-1} r^m \cos(m\theta). \tag{3-45}$$

Substituting the values for $S(r,\theta)$ and $C(r,\theta)$ (found in Appendix A) into n_k^i, reveals

$$\begin{aligned} \eta_k^i &= [\frac{A}{\sin(\theta)}\left(\frac{r\sin(\theta) + r^{k+1}\sin((k-1)\theta) - r^k \sin(k\theta)}{1 - 2r\cos(\theta) + r^2}\right) \\ &+ \left(\frac{1 - r\cos(\theta) + r^{k+1}\cos((k-1)\theta) - r^k\cos(k\theta)}{1 - 2r\cos(\theta) + r^2}\right) I] x_0 G_\beta \end{aligned} \tag{3-46}$$

The terms in the inhomogeneous solution, Eq. (3-46) that are independent of k

$$\left(\frac{A}{\sin(\theta)}\left(\frac{r\sin(\theta)}{\beta}\right) + \left(\frac{1 - r\cos(\theta)}{\beta}\right) I\right) x_0 G_\beta \tag{3-47}$$

simplify to

$$\frac{x_0}{\beta}\begin{bmatrix} \beta \\ 0 \end{bmatrix}, \tag{3-48}$$

or

$$x_0 U_1, \tag{3-49}$$

where

$$U_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3-50}$$

The inhomogeneous solution, Eq. (3-46), then becomes

$$\begin{aligned} \eta_k^i &= x_0 U_1 + \{\frac{1}{\beta}[\frac{A}{\sin(\theta)}\left(r^{k+1}\sin((k-1)\theta) - r^k \sin(k\theta)\right) \\ &+ \left(r^{k+1}\cos((k-1)\theta) - r^k \cos(k\theta)\right) I] x_0 G_\beta\}. \end{aligned} \tag{3-51}$$

Therefore, by substituting the simplified inhomogeneous solution Eq. (3-51) into Eq. (3-42), the general solution for the Constant Motion Model is of the form

$$\begin{aligned} \eta_k &= \frac{1}{\sin(\theta)}(r^k \sin((k-1)\theta) I + r^{k-1}\sin(k\theta) F_\beta)\eta_0 \\ &+ U_1 x_0 + \{\frac{1}{\beta}[\frac{A}{\sin(\theta)}\left(r^{k+1}\sin((k-1)\theta) - r^k \sin(k\theta)\right) \\ &+ \left(r^{k+1}\cos((k-1)\theta) - r^k \cos(k\theta)\right) I] x_0 G_\beta\}. \end{aligned} \tag{3-52}$$

Note that for α close to one, the filter converges quickly to steady state. While for smaller α, the filter converges very slowly to steady state.

The figures listed below of the Constant Motion Model illustrate the convergence properties of the filter.

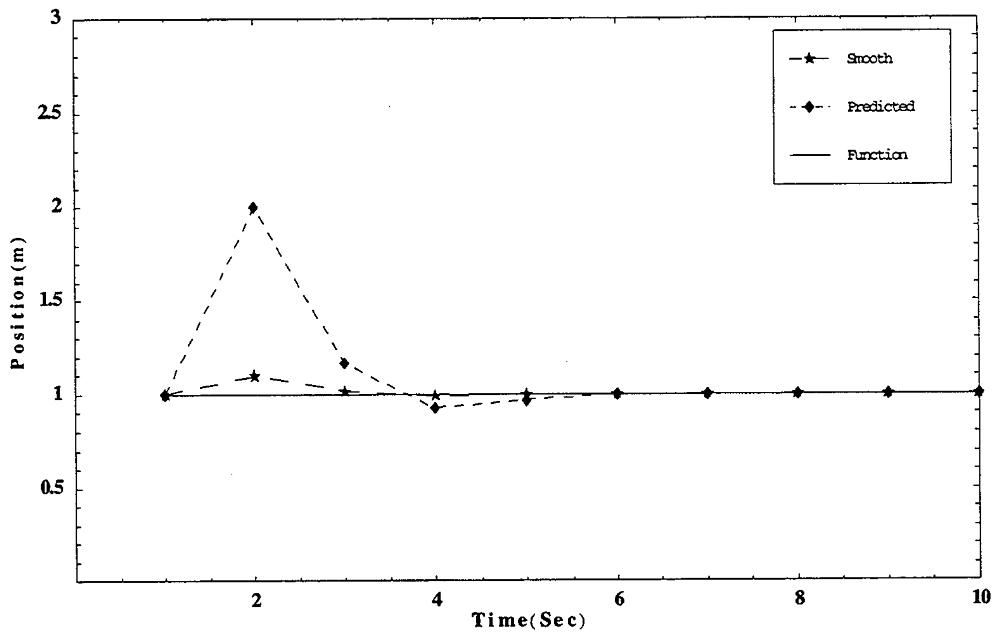

Figure 3.1-1. Constant Motion Model with $\alpha = .9$

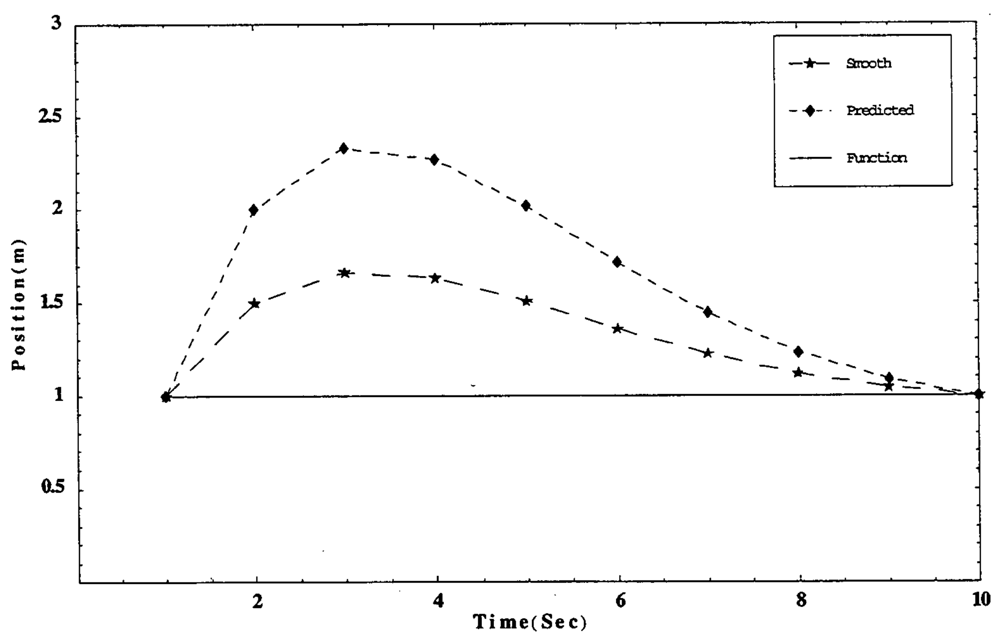

Figure 3.1-2. Constant Motion Model with $\alpha = .5$

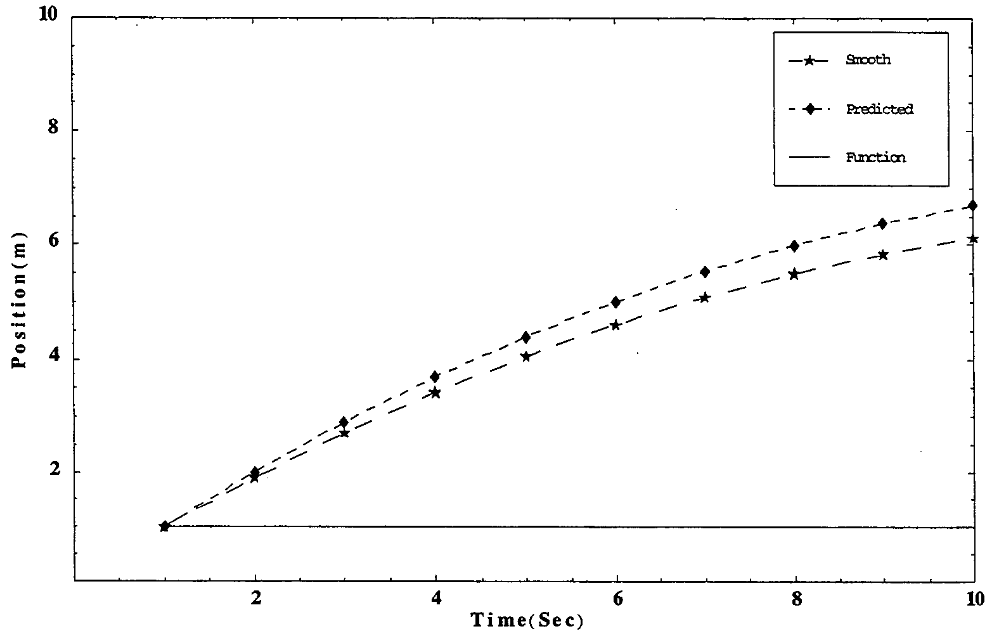

Figure 3.1-3. Constant Motion Model with $\alpha = .1$

3.2 LINEAR MOTION MODEL

If one assumes that x_m is linear, then x_m can be written as, $x_{k-m} = k - m$. Thus, the linear is separated into constant and linear terms. So one only needs to calculate the linear term and then apply the results gained from the constant term discussed previously in Section 3.1 to get the complete solution.

$$\begin{aligned} n_k &= \frac{1}{\sin(\theta)} \left[\left(-r^k \sin((k-1)\theta) \right) I + \left(r^{k-1} \sin(k\theta) \right) F_\beta \right] \eta_0 \\ &+ \left[\sum_{m=0}^{k-1} \left(\frac{A}{\sin(\theta)} r^m \sin(m\theta) + (r^m \cos(m\theta)) I \right) m \right] G_\beta \end{aligned} \tag{3-53}$$

Examine only the inhomogeneous solution

$$\eta_k^i = \left[\sum_{m=0}^{k-1} \left(\frac{A}{\sin(\theta)} r^m \sin(m\theta) + (r^m \cos(m\theta)) I \right) m \right] G_\beta \tag{3-54}$$

$$= \left[\frac{A}{\sin(\theta)}\sum_{m=0}^{k-1} mr^m \sin(m\theta) + \sum_{m=0}^{k-1}(mr^m\cos(m\theta))\,I\right] G_\beta$$

$$= \left[\frac{A}{\sin(\theta)} S_1(r,\theta) + C_1(r,\theta) I\right] G_\beta$$

where

$$S_1(r,\theta) = \sum_{m=0}^{k-1} mr^m \sin(m\theta) \tag{3-55}$$

and

$$C_1(r,\theta) = \sum_{m=0}^{k-1} mr^m \cos(m\theta). \tag{3-56}$$

Substituting the values for $S_1(r,\theta)$ and $C_1(r,\theta)$ (found in Appendix A) into n_k^i, reveals

$$\begin{aligned}
\eta_k^i &= \left[\frac{A}{\sin(\theta)} S_1(r,\theta) + C_1(r,\theta) I\right] G_\beta \qquad (3\text{-}57)\\
&= \frac{A}{\beta^2 \sin(\theta)}(\alpha r \sin(\theta) + \left((2-\alpha) r\cos(\theta) - 2r^2\right) I) G_\beta \\
&\quad +\{\frac{A}{\beta^2 \sin(\theta)}\left((\alpha + k\beta) r^{k+1}\sin((k-1)\theta) - (\alpha - \beta(1-k)) r^k \sin(k\theta)\right) \\
&\quad + \frac{1}{\beta^2}\left((\alpha + k\beta)\, r^{k+1}\cos((k-1)\theta) - (\alpha - \beta(1-k))\, r^k \cos(k\theta)\right)\} G_\beta
\end{aligned}$$

The terms in the inhomogeneous solution, Eq. (3-57), that are independence of k

$$\frac{1}{\beta^2}\left[\frac{A}{\sin(\theta)}\alpha r \sin(\theta) + \left((2-\alpha) r \cos(\theta) - 2r^2\right) I\right] G_\beta, \tag{3-58}$$

simplify to

$$\frac{1}{\beta^2}\begin{bmatrix} \alpha^2\beta - \alpha\beta + \alpha\beta - \alpha^2\beta \\ -\frac{\alpha^2\beta}{T} + \frac{\alpha^2\beta}{T} - \frac{\beta^2}{T} \end{bmatrix}, \tag{3-59}$$

or

$$\begin{bmatrix} 0 \\ -\frac{1}{T} \end{bmatrix}. \tag{3-60}$$

Let

$$L = \begin{bmatrix} 0 \\ -\frac{1}{T} \end{bmatrix}. \tag{3-61}$$

Therefore, by substituting the simplified inhomogeneous solution, Eq. (3-57) and the non-k dependent inhomogeneous solution, Eq. (3-61) into the general solution, Eq. (3-53)

$$\eta_k^i = L + \frac{1}{\beta^2}\{\frac{A}{\sin(\theta)}\left((\alpha + \beta k) r^{k+1} \sin((k-1)\theta) - (\alpha - \beta(1-k)) r^k \sin(k\theta)\right) \tag{3-62}$$

$$+\Big((\alpha+\beta k)\, r^{k+1}\cos((k-1)\theta) - (\alpha-\beta(1-k))\, r^{k}\cos(k\theta)\Big)\}G_{\beta}.$$

Combine the constant, k, with the linear term.

$$x_{k-m} = k - m. \tag{3-63}$$

Therefore, the Linear Motion Model solution is of the form

$$\begin{aligned}\eta_k \quad = \quad & \frac{1}{\sin(\theta)}(r^k \sin((k-1)\theta)I + r^{k-1}\sin(k\theta)F_{\beta})\eta_0 \\ & +(kI - L) \\ & +\{\frac{1}{\beta^2}[\frac{A}{\sin(\theta)}\Big(-\alpha r^{k+1}\sin((k-1)\theta) - (-\alpha+\beta)r^k \sin(k\theta)\Big) \\ & +\Big(-\alpha r^{k+1}\cos((k-1)\theta) - (-\alpha+\beta)r^k\cos(k\theta)\Big)\, I]x_0 G_{\beta}\}.\end{aligned} \tag{3-64}$$

Note that for α close to one, the filter converges quickly to steady state. While for smaller α, the filter converges very slowly to steady state.

The figures listed below of the Linear Motion Model illustrate the convergence properties of the filter.

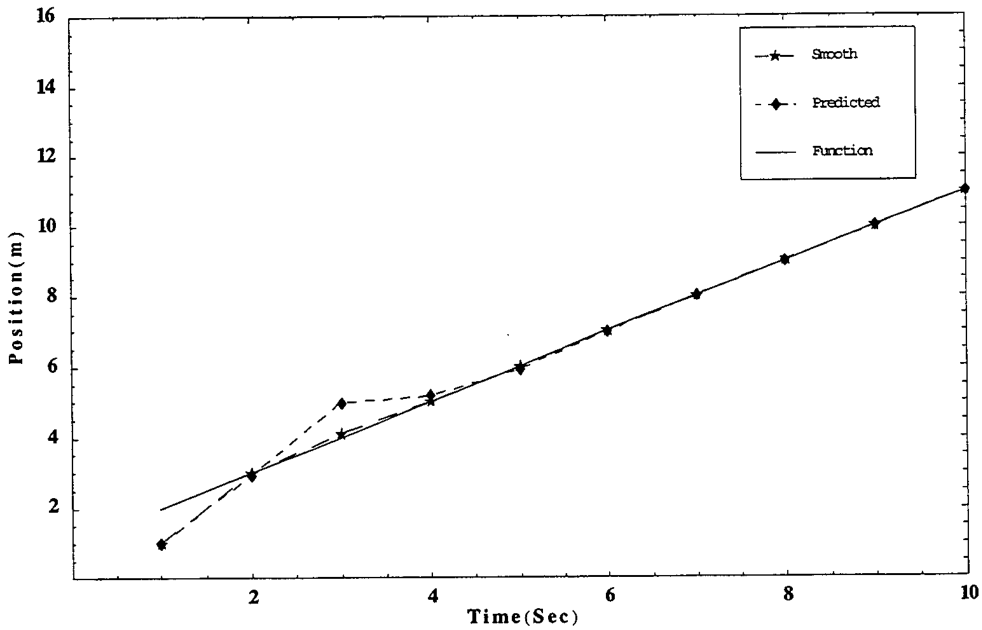

Figure 3.2-1. Linear Motion Model with $\alpha = .9$

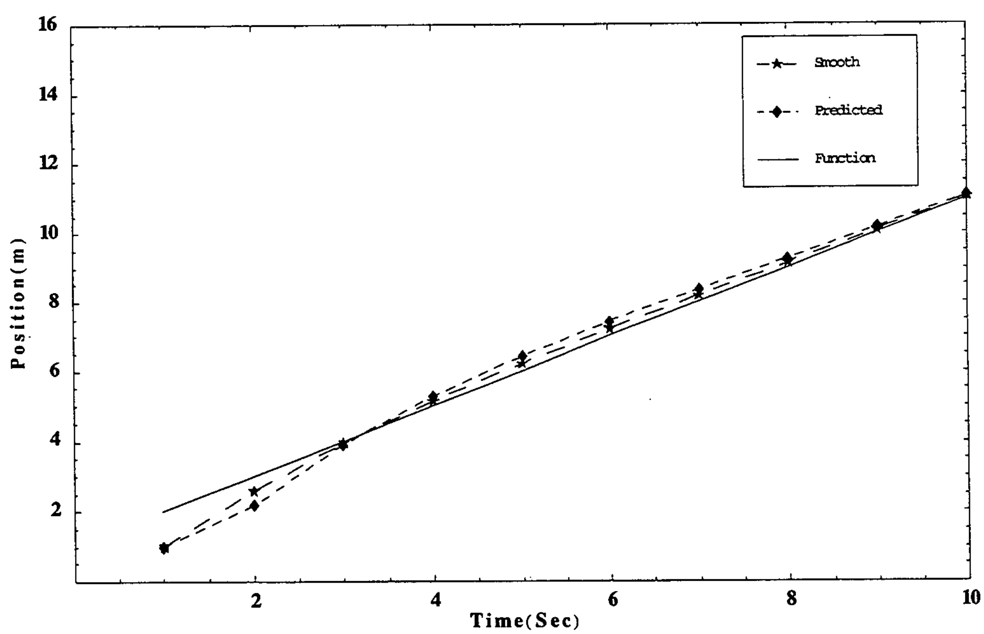

Figure 3.2-2. Linear Motion Model with $\alpha = .5$

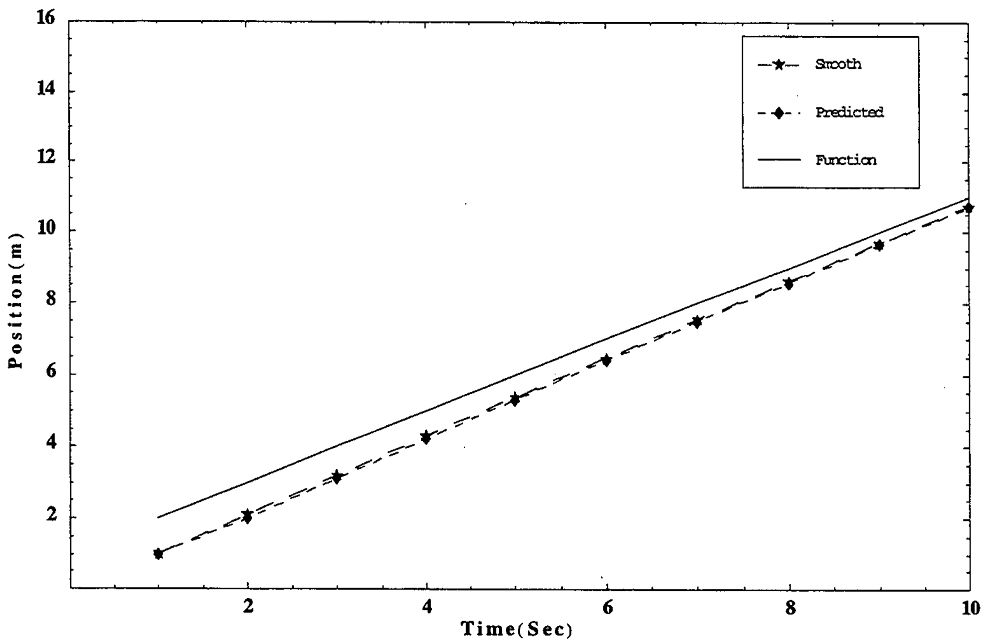

Figure 3.2-3. Linear Motion Model with $\alpha = .1$

3.3 QUADRATIC MOTION MODEL

If one assumes that x_m is quadratic, then x_m can be written as $x_{k-m} = (k-m)^2 = (k^2 - 2km + m^2)$. Thus, the quadratic is separated into constant, linear, and quadratic terms. So one only needs to calculate the quadratic term and then apply the results gained from the constant and linear terms discussed previously to get the complete solution.

$$\begin{aligned} n_k &= \frac{1}{\sin(\theta)}\left[\left(-r^k \sin((k-1)\theta)\right) I + \left(r^{k-1}\sin(k\theta)\right) F_\beta\right]\eta_0 \\ &+ \left[\sum_{m=0}^{k-1}\left(\frac{A}{\sin(\theta)}r^m \sin(m\theta) + (r^m\cos(m\theta))\, I\right) m^2\right] G_\beta \end{aligned} \tag{3-65}$$

Consider only the inhomogeneous solution of Eq. (3-65),

$$\eta_k^i = \left[\sum_{m=0}^{k-1}\left(\frac{A}{\sin(\theta)}r^m \sin(m\theta) + (r^m\cos(m\theta))\, I\right) m^2\right] G_\beta \tag{3-66}$$

$$= \left[\frac{A}{\sin(\theta)}\sum_{m=0}^{k-1} r^m \sin(m\theta)m^2 + \sum_{m=0}^{k-1}\left(r^m \cos(m\theta)\right)m^2 I\right] G_\beta$$

$$= \left[\frac{A}{\sin(\theta)}\sum_{m=0}^{k-1} m^2 r^m \sin(m\theta) + \sum_{m=0}^{k-1}\left(m^2 r^m \cos(m\theta)\right) I\right] G_\beta$$

$$= \left[\frac{A}{\sin(\theta)} S_2(r,\theta) + C_2(r,\theta) I\right] G_\beta$$

where

$$S_2(r,\theta) = \sum_{m=0}^{k-1} m^2 r^m \sin(m\theta) \tag{3-67}$$

and

$$C_2(r,\theta) = \sum_{m=0}^{k-1} m^2 r^m \cos(m\theta). \tag{3-68}$$

Substituting the value of $S_2(r,\theta)$ and $C_2(r,\theta)$ (found in Appendix A) into n_k^i, reveals

$$\begin{aligned}
\eta_k^i &= \left[\frac{A}{\sin(\theta)} S_2(r,\theta) + C_2(r,\theta) I\right] G_\beta \qquad (3\text{-}69) \\
&= \frac{1}{\beta^3}\left(\frac{A}{\sin(\theta)}(2\alpha^2 + \alpha\beta - 2\beta) r \sin(\theta) - 4\alpha r^2 I + (4\alpha - 2\alpha^2 - \alpha\beta) r \cos(\theta) I\right) G_\beta \\
&+ \frac{A}{\beta^3 \sin(\theta)}\{(2\alpha^2 - 2\beta + \alpha\beta + 2\alpha\beta k + k^2\beta^2) r^{k+1} \sin((k-1)\theta) \\
&-(2\alpha^2 - 2\beta - \alpha\beta + 2\alpha\beta k + \beta^2 - 2k\beta^2 + k^2\beta^2) r^k \sin(k\theta)\} \\
&+ \frac{1}{\beta^3}\{(2\alpha^2 - 2\beta + \alpha\beta + 2\alpha\beta k + k^2\beta^2) r^{k+1} \cos((k-1)\theta) \\
&-(2\alpha^2 - 2\beta - \alpha\beta + 2\alpha\beta k + \beta^2 - 2k\beta^2 + k^2\beta^2) r^k \cos(k\theta)\}\} G_\beta
\end{aligned}$$

The terms of the inhomogeneous solution, Eq. (3-69), that are independent of k

$$\frac{1}{\beta^3}\left(\frac{A}{\sin(\theta)}(2\alpha^2 + \alpha\beta - 2\beta) r \sin(\theta) - 4\alpha r^2 I + (4\alpha - 2\alpha^2 - \alpha\beta) r \cos(\theta) I\right) G_\beta, \tag{3-70}$$

which simplify to

$$\frac{1}{\beta^3}\begin{bmatrix} 2\beta^2(\alpha - 1) \\ \beta^2(2\alpha - \beta) \end{bmatrix}, \tag{3-71}$$

or

$$\begin{bmatrix} \frac{2(\alpha-1)}{\beta} \\ \frac{-(2\alpha-\beta)}{T\beta} \end{bmatrix}. \tag{3-72}$$

Let

$$M = \begin{bmatrix} \frac{2(\alpha-1)}{\beta} \\ \frac{-(2\alpha-\beta)}{T\beta} \end{bmatrix}, \tag{3-73}$$

then the inhomogeneous solution, Eq. (3-69), becomes

$$\begin{aligned} \eta_k^i &= M + \frac{1}{\beta^3}\{\frac{A}{\sin(\theta)}\{(2\alpha^2 - 2\beta + \alpha\beta + 2\alpha\beta k + k^2\beta^2)r^{k+1}\sin((k-1)\theta) \\ &-(2\alpha^2 - 2\beta - \alpha\beta + 2\alpha\beta k + \beta^2 - 2k\beta^2 + k^2\beta^2)r^k \sin(k\theta)\} \\ &+\{(2\alpha^2 - 2\beta + \alpha\beta + 2\alpha\beta k + k^2\beta^2)r^{k+1}\cos((k-1)\theta) \\ &-(2\alpha^2 - 2\beta - \alpha\beta + 2\alpha\beta k + \beta^2 - 2k\beta^2 + k^2\beta^2)r^k \cos(k\theta)\}I\}G_\beta \end{aligned} \tag{3-74}$$

Combine the constant, k^2, and linear,$-2km$, terms with the quadratic term

$$x_{k-m} = k^2 - 2km + m^2. \tag{3-75}$$

Therefore, the Quadratic Motion Model solution is of the form

$$\begin{aligned} n_k &= \frac{1}{\sin(\theta)}\left[\left(-r^k \sin((k-1)\theta)\right) I + \left(r^{k-1}\sin(k\theta)\right) F_\beta\right] \eta_0 \\ &+[(k^2 I - 2kL + M) \\ &+\frac{1}{\beta^3}\{\frac{A}{\sin(\theta)}[\left(2\alpha^2 - 2\beta + \alpha\beta\right) r^{k+1}\sin((k-1)\theta) \\ &- \left(2\alpha^2 - 2\beta - \alpha\beta + \beta^2\right) r^k \sin(k\theta)] \\ &+ \left(2\alpha^2 - 2\beta + \alpha\beta\right) r^k \cos((k-1)\theta) I \\ &- \left(2\alpha^2 - 2\beta - \alpha\beta + \beta^2\right) r^k \cos(k\theta) I\}G_\beta \end{aligned} \tag{3-76}$$

The terms in the general solution of the Quadratic Motion Model, Eq. (3-76) that are not transient and dependent on m

$$k^2 I - 2kL + M \tag{3-77}$$

simplify to

$$\left(k^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2k \begin{bmatrix} 0 \\ -\frac{1}{T} \end{bmatrix} + \begin{bmatrix} \frac{-2(1-\alpha)}{\beta} \\ -2\left(\frac{\alpha}{\beta} - \frac{1}{2}\right) \end{bmatrix}\right), \tag{3-78}$$

or

$$\begin{bmatrix} k^2 - 2\left(\frac{1-\alpha}{\beta}\right) \\ 2k - 2\left(\frac{\alpha}{\beta} - \frac{1}{2}\right) \end{bmatrix}. \tag{3-79}$$

Recall

$$n_k^i = \begin{bmatrix} x_s \\ v_s \end{bmatrix}, \tag{3-80}$$

so one can define

$$l_p = \left(\frac{1-\alpha}{\beta}\right) \tag{3-81}$$

where l_p is the position lag in the response due to the acceleration input, and

$$l_v = \left(\frac{\alpha}{\beta} - \frac{1}{2}\right) \tag{3-82}$$

where l_v is the velocity lag in the response due to the acceleration input [14]. The lags mentioned above are graphed in Figure 3.3-1.

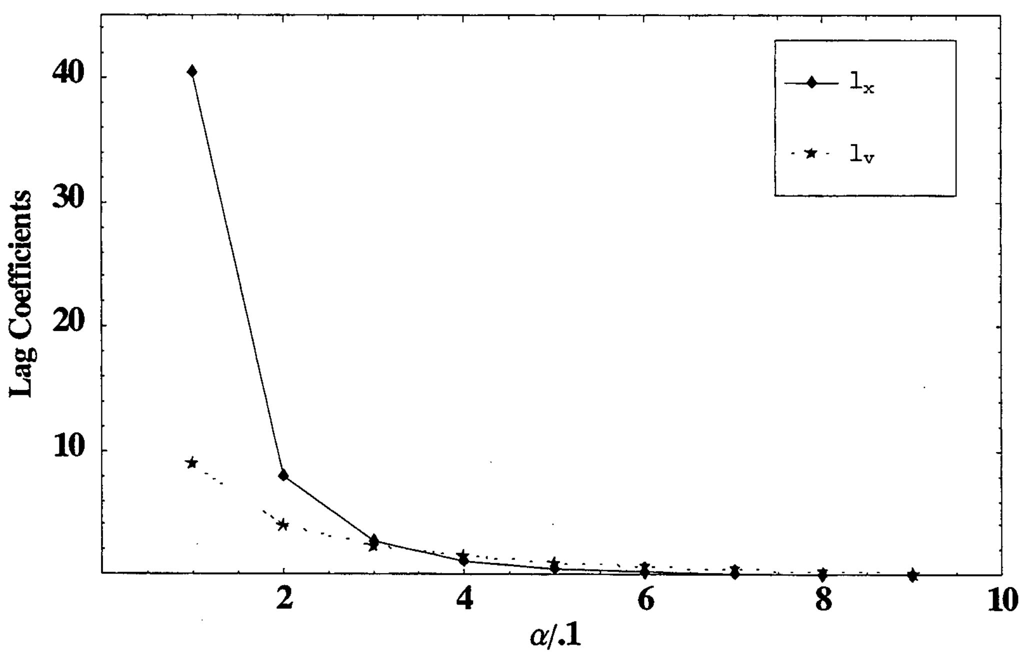

Figure 3.3-1. Steady-state position and velocity lags

Since $0 < r < 1$, the terms r^k and r^{k+1} represent the exponential damping in the transient response. Therefore, the non-transient portion of the smoothed position solution is

$$x_{ss} = \frac{a_0}{2}T^2k^2 - \frac{a_0 l_p T^2}{2}, \tag{3-83}$$

and the non-transient portion of the smoothed velocity solution is

$$v_{ss} = a_0 T k - 2a_0 l_v T. \tag{3-84}$$

The figures listed below of the Quadratic Motion Model illustrate the convergence properties of the filter.

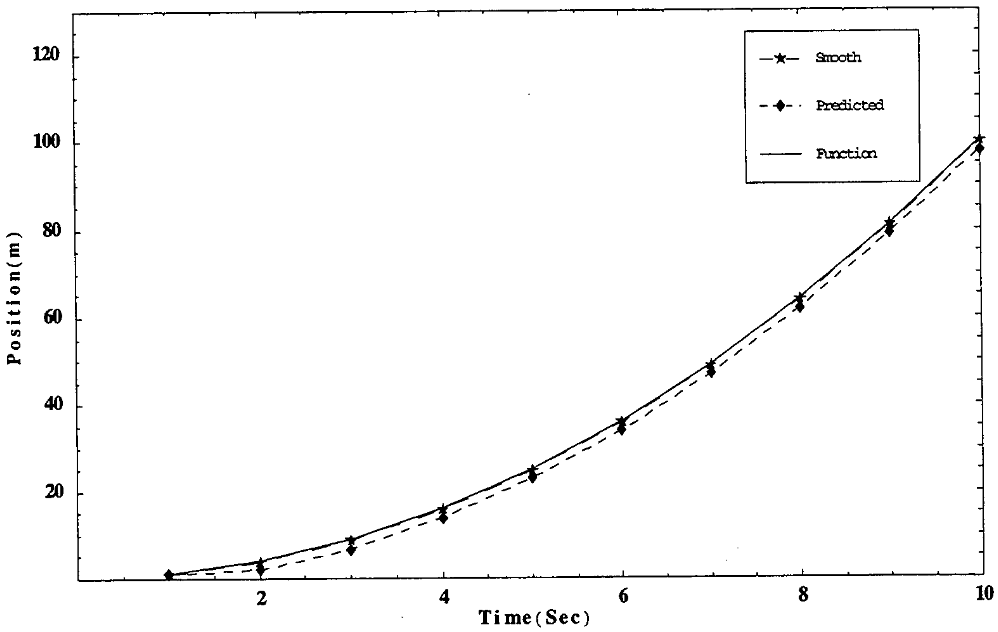

Figure 3.3-2. Quadratic Motion Model with $\alpha = .9$

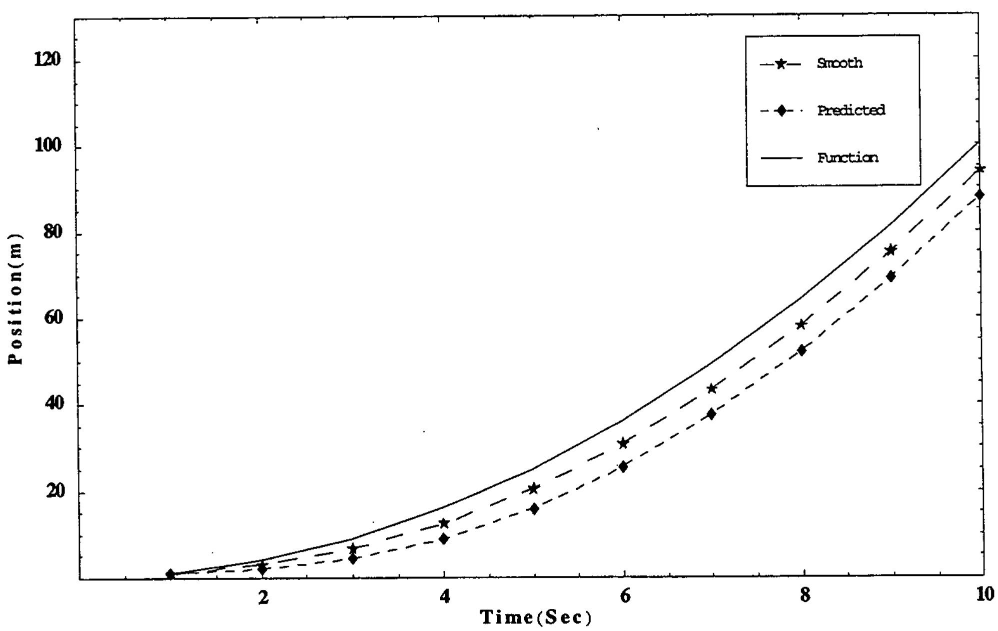

Figure 3.3-3. Quadratic Motion Model with $\alpha = .5$

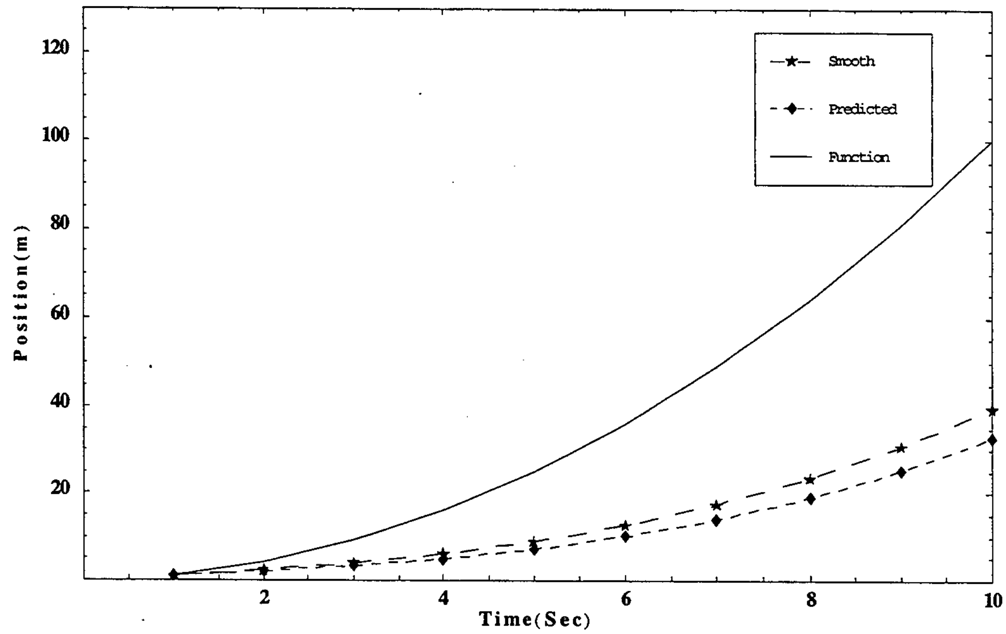

Figure 3.3-4. Quadratic Motion Model with $\alpha = .1$

3.4 SINUSOIDAL MOTION MODEL

If one assumes that x_m is sinusoidal, then x_m can be written as, $x_{k-m} = \cos(k-m)$, then the general solution for the Sinusoidal Motion Model becomes

$$\begin{aligned} n_k &= \frac{1}{\sin(\theta)}\left[\left(-r^k \sin((k-1)\theta)\right) I + \left(r^{k-1}\sin(k\theta)\right) F_\beta\right]\eta_0 \\ &+ \left[\sum_{m=0}^{k-1}\left(\frac{A}{\sin(\theta)} r^m \sin(m\theta) + (r^m \cos(m\theta))\, I\right)\cos(k-m)\right] G_\beta \end{aligned} \tag{3-85}$$

Note that for α close to one, the filter converges quickly to steady state. While for smaller α, the filter converges very slowly to steady state.

The figures listed below of the Sinusoidal Motion Model illustrate the convergence properties of the filter.

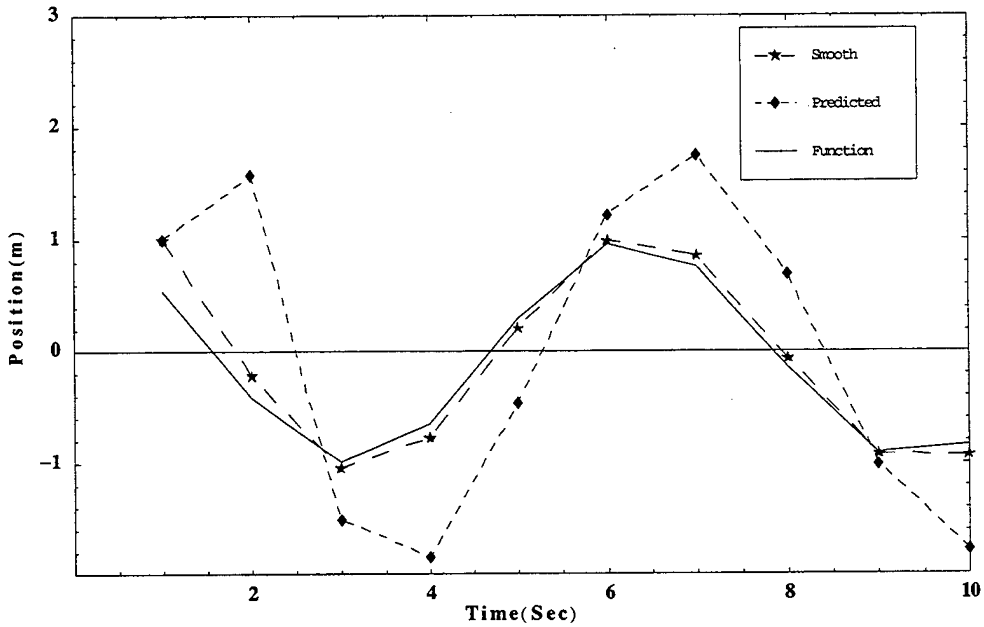

Figure 3.4-1. Sinusoidal Motion Model with $\alpha = .9$

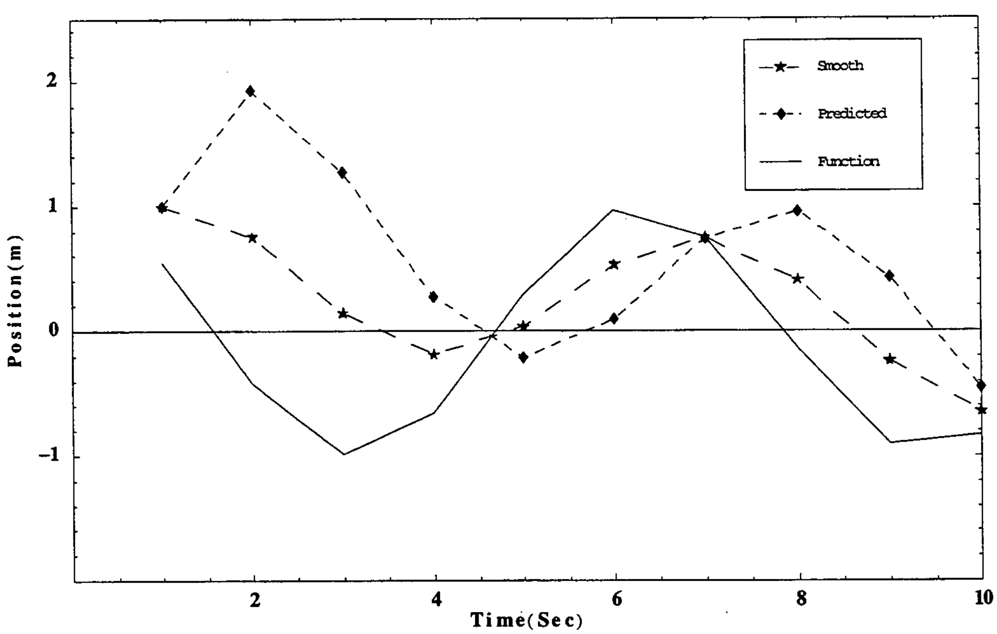

Figure 3.4-2. Sinusoidal Motion Model with $\alpha = .5$

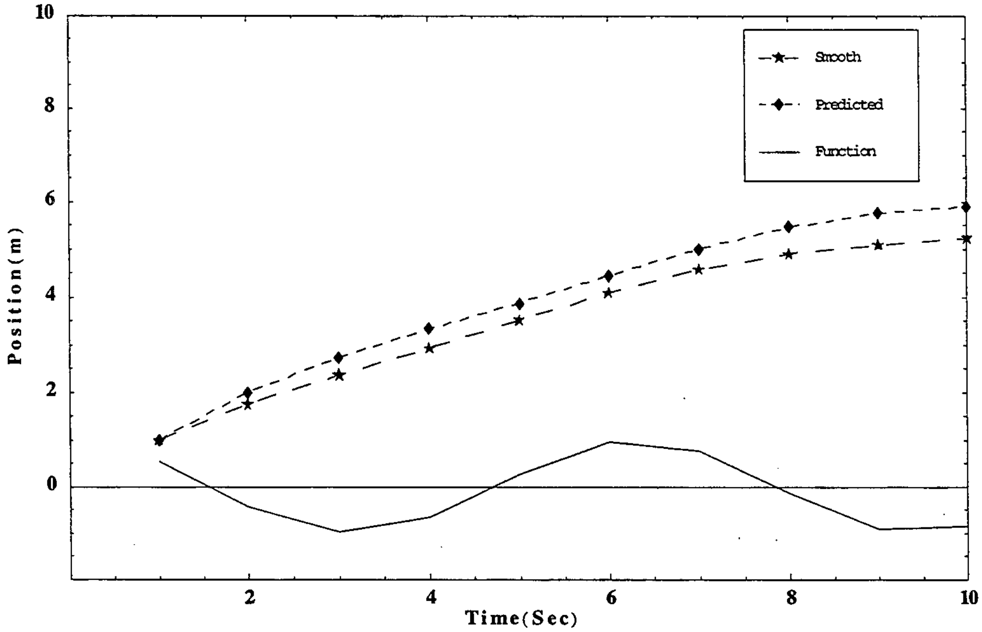

Figure 3.4-3. Sinusoidal Motion Model with $\alpha = .1$

4 $\alpha - \beta - \gamma$ FILTER

The tracking equations for the $\alpha - \beta - \gamma$ filter consists of two parts: prediction equations, which are given by

$$x_p(k) = x_s(k-1) + v_s(k-1)T + \frac{T^2}{2}a_s(k-1) \tag{4-1}$$

$$v_p(k) = v_s(k-1) + Ta_s(k-1) \tag{4-2}$$

$$a_p(k) = a_s(k-1) \tag{4-3}$$

and smoothing equations, which are given by

$$x_s(k) = x_p(k) + \alpha(x_m(k) - x_p(k)) \tag{4-4}$$

$$v_s(k) = v_p(k) + \frac{\beta}{T}(x_m(k) - x_p(k)) \tag{4-5}$$

$$a_s(k) = a_p(k) + \frac{\gamma}{T^2}(x_m(k) - x_p(k)) \tag{4-6}$$

where

- $x_s(k)$ = smoothed position at the k-th interval
- $x_p(k)$ = predicted position at the k-th interval
- $x_m(k)$ = measured position at the k-th interval
- $v_s(k)$ = smoothed velocity at the k-th interval
- $v_p(k)$ = predicted velocity at the k-th interval
- $a_s(k)$ = smoothed acceleration at the k-th interval
- $a_p(k)$ = predicted acceleration at the k-th interval
- T = radar update interval or period
- α, β, γ = filter weighing coefficients

For the $\alpha - \beta - \gamma$, the commonly used relationship between α and β is the Kalata relationship which is obtained from steady state Kalman filter theory assuming zero mean white noise in the position and velocity state equations [13]. An additional relation between γ and $\alpha - \beta$ is needed. The most common is

$$\gamma = \frac{\beta^2}{2\alpha} \tag{4-7}$$

which is known as the Neal-Simpson relation[13].

Alternatively, Eq. (4-1) through Eq. (4-6) can be written as

$$|x_s\rangle_k = H_\gamma \, |x_p\rangle_k + G_\gamma x_m(k) \tag{4-8}$$

and

$$|x_p\rangle_{k+1} = Q_\gamma \, |x_s\rangle_k \tag{4-9}$$

where

$$|x_s\rangle_k = \begin{bmatrix} x_s(k) \\ v_s(k) \\ a_s(k) \end{bmatrix} \tag{4-10}$$

$$|x_p\rangle_k = \begin{bmatrix} x_p(k) \\ v_p(k) \\ a_p(k) \end{bmatrix} \tag{4-11}$$

$$Q_\gamma = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{4-12}$$

$$H_\gamma = \begin{bmatrix} 1-\alpha & 0 & 0 \\ -\frac{\beta}{T} & 1 & 0 \\ -\frac{\gamma}{T^2} & 0 & 1 \end{bmatrix} \tag{4-13}$$

$$G_\gamma = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \\ \frac{\gamma}{T^2} \end{bmatrix} \tag{4-14}$$

Alternatively, Eq. (4-8) can be written as

$$|x_s\rangle_k = F_\gamma \, |x_s\rangle_{k-1} + G_\gamma x_m(k) \tag{4-15}$$

where

$$F_\gamma = H_\gamma \cdot Q_\gamma = \begin{bmatrix} 1-\alpha & (1-\alpha)\,T & (1-\alpha)\,\frac{T^2}{2} \\ -\frac{\beta}{T} & 1-\beta & \left(1-\frac{\beta}{2}\right)T \\ -\frac{\gamma}{T^2} & -\frac{\gamma}{T} & 1-\frac{\gamma}{2} \end{bmatrix} \tag{4-16}$$

while the G matrix is

$$G_\gamma = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \\ \frac{\gamma}{T^2} \end{bmatrix}. \tag{4-17}$$

The eigenvalues of F_γ can be shown to satisfy the equation (using the result in Eq. (4-16) and expanding about the 33−minor)

$$f(\lambda) = 0 = (\lambda - re^{j\theta})(\lambda - re^{-j\theta})(1 - \frac{\gamma}{2} - \lambda) + \frac{\gamma\lambda}{2}(3 - \beta - \alpha) + \frac{(1-\alpha)\,\gamma}{2}, \tag{4-18}$$

or

$$f(\lambda) = 0 = (\lambda^2 - 2r\cos(\theta) + r^2)(1 - \frac{\gamma}{2} - \lambda) - \frac{\gamma\lambda}{2}(1 + 2r\cos(\theta)) - \frac{(1-\alpha)\,\gamma}{2}. \tag{4-19}$$

which simplifies to

$$f(\lambda) = 0 = \lambda^3 + \lambda^2(-1 + \frac{\gamma}{2} - 2r\cos(\theta)) + \lambda(r^2 + \frac{\gamma}{2} + 2r\cos(\theta)) - r^2 \qquad (4\text{-}20)$$

$$f(\lambda) = 0 = \lambda^3 - \lambda^2 b + \lambda\left(b + r^2 - s^2\right) - r^2 \qquad (4\text{-}21)$$

where $\gamma = 1 - s^2$ and $\left(\frac{s^2}{2} + \frac{1}{2} + 2r\cos(\theta)\right) = b$. Without loss of generality, one can assume that Eq. (4-21) can be written as

$$\begin{aligned} f(\lambda) &= 0 = \left(\lambda - te^{j\varphi}\right)\left(\lambda - te^{-j\varphi}\right)(\lambda - q) \\ &= 0 = (\lambda^2 - 2t\lambda\cos(\varphi) + t^2)(\lambda - q). \end{aligned} \qquad (4\text{-}22)$$

Equating Eq. (4-21) to Eq. (4-22) leads to three equations in three unknowns:

$$2t\cos\varphi + q = b \qquad (4\text{-}23)$$

$$t^2 + 2tq\cos\varphi = b + r^2 - s^2 \qquad (4\text{-}24)$$

$$qt^2 = r^2 \qquad (4\text{-}25)$$

Eq. (4-23) and Eq. (4-25) have solutions

$$2t\cos\varphi = b - q \qquad (4\text{-}26)$$

$$q = \frac{r^2}{t^2} \qquad (4\text{-}27)$$

while the Eq. (4-24) leads to the equation

$$t^6 - \left(b + r^2 - s^2\right)t^4 + br^2t^2 - r^4 = 0, \qquad (4\text{-}28)$$

which is a cubic equation in t^2. Making the substitution $y = t^2 - \frac{b + r^2 - s^2}{3}$, reduces Eq. (4-28) to the fundamental form

$$y^3 + py + m = 0, \qquad (4\text{-}29)$$

where

$$p = -\frac{1}{3}\left(b + r^2 - s^2\right)^2 + br^2, \qquad (4\text{-}30)$$

$$m = -\frac{2}{27}\left(b + r^2 - s^2\right)^3 + \frac{br^2}{3}\left(b + r^2 - s^2\right) - r^4. \qquad (4\text{-}31)$$

Find u and v so that

$$3uv = p, \qquad (4\text{-}32)$$

and

$$u^3 - v^3 = m. \qquad (4\text{-}33)$$

By solving for u in Eq. (4-32) and substituting into Eq. (4-33), the result is given as

$$\left(\frac{p}{3v}\right)^3 - v^3 = m \tag{4-34}$$

which simplifies to

$$v^6 + mv^3 - \frac{p^3}{27} = 0. \tag{4-35}$$

which by the quadratic formula, has the solution

$$v^3 = \frac{-m \pm \sqrt{m^2 + \frac{4p^3}{27}}}{2}. \tag{4-36}$$

To continue, one needs to find the value of u by substituting the value of v into Eq. (4-33)

$$u^3 = m + \frac{-m \pm \sqrt{m^2 + \frac{4p^3}{27}}}{2} \tag{4-37}$$

Hence, the value of y can be found by subtracting the values of u and v to give the following:

$$y = \sqrt[3]{\frac{m \pm \sqrt{m^2 + \frac{4p^3}{27}}}{2}} - \sqrt[3]{\frac{-m \pm \sqrt{m^2 + \frac{4p^3}{27}}}{2}} \tag{4-38}$$

Therefore, the value of t^2 can be found by substituting the value of y into $y = t^2 - \frac{b + r^2 - s^2}{3}$,

$$t^2 = \sqrt[3]{\frac{m \pm \sqrt{m^2 + \frac{4p^3}{27}}}{2}} - \sqrt[3]{\frac{-m \pm \sqrt{m^2 + \frac{4p^3}{27}}}{2}} + \frac{b + r^2 - s^2}{3} \tag{4-39}$$

It follows that the values of q and φ are now known.

Let $h(\lambda) = \lambda^k$ and $g(\lambda) = \gamma_0(k) + \gamma_1(k)\lambda + \gamma_2(k)\lambda^2$. By equating $h(\lambda)$ and $g(\lambda)$, one can see that if

$$h(\lambda_0) = g(\lambda_0) \tag{4-40}$$

then

$$t^k e^{jk\varphi} = \gamma_0(k) + \gamma_1(k)te^{j\varphi} + \gamma_2(k)t^2e^{2j\varphi}, \tag{4-41}$$

if

$$h(\lambda_1) = g(\lambda_1) \tag{4-42}$$

then

$$t^k e^{-jk\varphi} = \gamma_0(k) + \gamma_1(k)te^{-j\varphi} + \gamma_2(k)t^2e^{-2j\varphi}, \tag{4-43}$$

and if

$$h(\lambda_2) = g(\lambda_2) \tag{4-44}$$

then

$$q^k = \gamma_0(k) + \gamma_1(k)q + \gamma_2(k)q^2. \tag{4-45}$$

Hence, there exists three simultaneous equations to solve for the three eigenvalues to get the coefficients[5]:

$$t^k e^{jk\varphi} = \gamma_0(k) + \gamma_1(k) t e^{j\varphi} + \gamma_2(k) t^2 e^{2j\varphi}, \tag{4-46}$$

$$t^k e^{-jk\varphi} = \gamma_0(k) + \gamma_1(k) t e^{-j\varphi} + \gamma_2(k) t^2 e^{-2j\varphi}, \tag{4-47}$$

and

$$q^k = \gamma_0(k) + \gamma_1(k) q + \gamma_2(k) q^2. \tag{4-48}$$

Solving these equations, one obtains

$$\gamma_0(k) = \xi \left(q t^{k+1} \sin((k-2)\varphi) - q^2 t^k \sin((k-1)\varphi) + q^k t^2 \sin(\varphi) \right), \tag{4-49}$$

$$\gamma_1(k) = \xi \left(-t^{k+1} \sin((k-2)\varphi) + q t^{k-1} \sin(\varphi) + q^k t \sin(2\varphi) \right), \tag{4-50}$$

and

$$\gamma_2(k) = \xi \left(t^k \sin((k-1)\varphi) - q t^{k-1} \sin(k\varphi) + q^k \sin(\varphi) \right), \tag{4-51}$$

where

$$\xi = \frac{1}{\sin(\varphi)[-2qt\cos(\varphi) + q^2 + t^2]} \tag{4-52}$$

The general solution to the $\alpha - \beta - \gamma$ filter equations consists of the homogeneous solution

$$\eta_k^h = F^k \eta_0, \tag{4-53}$$

where $F^k = \gamma_0(k) I + \gamma_1(k) F_\gamma + \gamma_2(k) F_\gamma^2$, hence

$$\eta_k^h = \left[\gamma_0(k) I + \gamma_1(k) F_\gamma + \gamma_2(k) F_\gamma^2 \right] \eta_0. \tag{4-54}$$

By substituting Eq. (4-49) through Eq. (4-52) into Eq. (4-54), the homogeneous solution can be written as follows:

$$\begin{aligned} \eta_k^h \quad = \quad & \xi \{ \left(q t^{k+1} \sin((k-2)\varphi) - q^2 t^k \sin((k-1)\varphi) + q^k t^2 \sin(\varphi) \right) I \\ & + \left(-t^{k+1} \sin((k-2)\varphi) + q t^{k-1} \sin(\varphi) + q^k t \sin(2\varphi) \right) F_\gamma \\ & + \left(t^k \sin((k-1)\varphi) - q t^{k-1} \sin(k\varphi) + q^k \sin(\varphi) \right) F_\gamma^2 \} \eta_0. \end{aligned} \tag{4-55}$$

Recall, the inhomogeneous solution is of the form

$$\eta_k^i = \sum_{n=1}^{k} F^{k-n} x_n G_\gamma, \tag{4-56}$$

where

$$F^{k-n} = \gamma_0(k-n) I + \gamma_1(k-n) F_\gamma + \gamma_2(k-n) F_\gamma^2 \tag{4-57}$$

hence,

$$\begin{aligned}\eta_k^i &= \sum_{n=1}^{k}[\gamma_0(k-n)I+\gamma_1(k-n)F_\gamma+\gamma_2(k-n)F_\gamma^2]x_nG_\gamma \\ &= \sum_{m=0}^{k-1}[\gamma_0(m)I+\gamma_1(m)F_\gamma+\gamma_2(m)F_\gamma^2]x_{k-m}G_\gamma.\end{aligned} \tag{4-58}$$

By substituting Eq. (4-49) through Eq. (4-52) into Eq. (4-58), the inhomogeneous solutions can be written as follows:

$$\begin{aligned}\eta_k^i &= \xi\sum_{m=0}^{k-1}\{\left(qt^{m+1}\sin((m-2)\varphi)-q^2t^m\sin((m-1)\varphi)+q^mt^2\sin(\varphi)\right)I \\ &+\left(-t^{m+1}\sin((m-2)\varphi)+qt^{m-1}\sin(\varphi)+q^mt\sin(2\varphi)\right)F_\gamma \\ &+\left(t^m\sin((m-1)\varphi)-qt^{m-1}\sin(m\varphi)+q^m\sin(\varphi)\right)F_\gamma^2\}x_{k-m}G_\gamma.\end{aligned} \tag{4-59}$$

One can further simplify the inhomogeneous solutions in Eq. (4-59) by using the trigonometric identities

$$\sin[(m-2)\varphi]=(\cos^2[\varphi]-\sin^2[\varphi])\sin[m\varphi]-(2\sin[\varphi]\cos[\varphi])\cos[m\varphi] \tag{4-60}$$

and

$$\sin[(m-1)\varphi]=\sin(m\varphi)\cos(\varphi)-\cos(m\varphi)\sin(\varphi), \tag{4-61}$$

therefore

$$\eta_k^i=\sum_{m=0}^{k-1}\left[\left((Bt^m\sin(m\theta)+(Dt^m\cos(m\theta))+E\right)x_{k-m}\right]G_\beta, \tag{4-62}$$

where

$$B=-q(q\cos[\varphi]-t\cos[2\varphi])I-t\cos[2\varphi]F_\gamma+t(-q+t\cos[\varphi])F_\gamma^2, \tag{4-63}$$

$$D=\sin[\varphi]\{q(q-2t\cos[\varphi])I+2t\cos[\varphi]F_\gamma-F_\gamma^2\}, \tag{4-64}$$

and

$$E=\sin[\varphi]\{q^mt^2I+qt^{m-1}(-1+2qt\cos[\varphi])F_\gamma+q^mF_\gamma^2\}. \tag{4-65}$$

Therefore, the combination of homogeneous, Eq. (4-55), and inhomogeneous, Eq. (4-62), solutions lends the general solution in the following form:

$$\begin{aligned}n_k &= \xi\{\{\left(qt^{k+1}\sin((k-2)\varphi)-q^2t^k\sin((k-1)\varphi)+q^kt^2\sin(\varphi)\right)I \\ &+\left(-t^{k+1}\sin((k-2)\varphi)+qt^{k-1}\sin(\varphi)+q^kt\sin(2\varphi)\right)F_\gamma \\ &+\left(t^k\sin((k-1)\varphi)-qt^{k-1}\sin(k\varphi)+q^k\sin(\varphi)\right)F_\gamma^2\}\eta_0\end{aligned} \tag{4-66}$$

$$+\sum_{m=0}^{k-1}\left[\left(\left(Bt^m\sin(m\varphi)+\left(Dt^m\cos(m\varphi)\right)+E\right)x_{k-m}\right]G_\beta.$$

4.1 CONSTANT ACCELERATION MOTION MODEL

If one assumes that $x_m = x_0 + v_0 m + \frac{1}{2}a_0 m^2$, then the general solution for the Constant Acceleration Motion Model is of the form

$$\begin{aligned} n_k &= \xi\{\{\left(qt^{k+1}\sin((k-2)\varphi) - q^2t^k\sin((k-1)\varphi) + q^kt^2\sin(\varphi)\right)I \\ &+ \left(-t^{k+1}\sin((k-2)\varphi) + qt^{k-1}\sin(\varphi) + q^kt\sin(2\varphi)\right)F_\gamma \\ &+ \left(t^k\sin((k-1)\varphi) - qt^{k-1}\sin(k\varphi) + q^k\sin(\varphi)\right)F_\gamma^2\}\eta_0 \\ &+ \sum_{m=0}^{k-1}\left[\left(\left(Bt^m\sin(m\varphi)+\left(Dt^m\cos(m\varphi)\right)+E\right)\left(x_0+v_0m+\frac{1}{2}a_0m^2\right)\right]G_\beta. \end{aligned} \tag{4-67}$$

The figures listed below of the Constant Motion Model illustrate the convergence properties of the filter. Note that for α close to one, the filter converges quickly to steady state. While for smaller α, the filter converges very slowly to steady state.

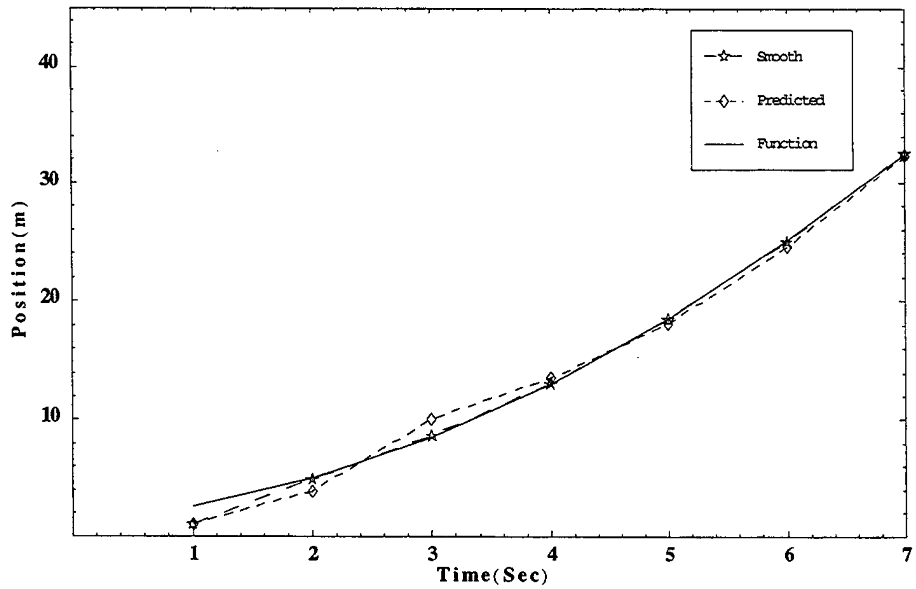

Figure 4.1-1. Constant Acceleration Motion Model with $\alpha = .9$

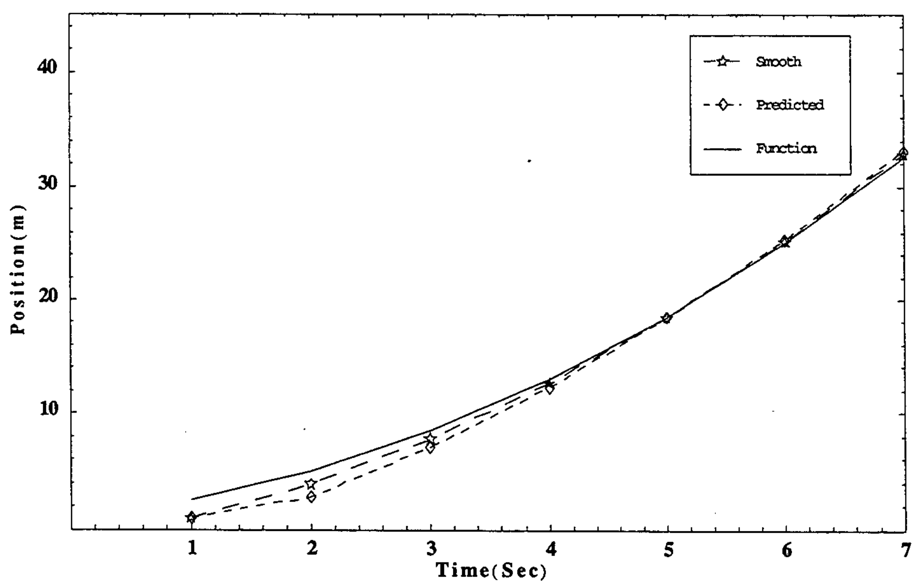

Figure 4.1-2. Constant Acceleration Motion Model with $\alpha = .5$

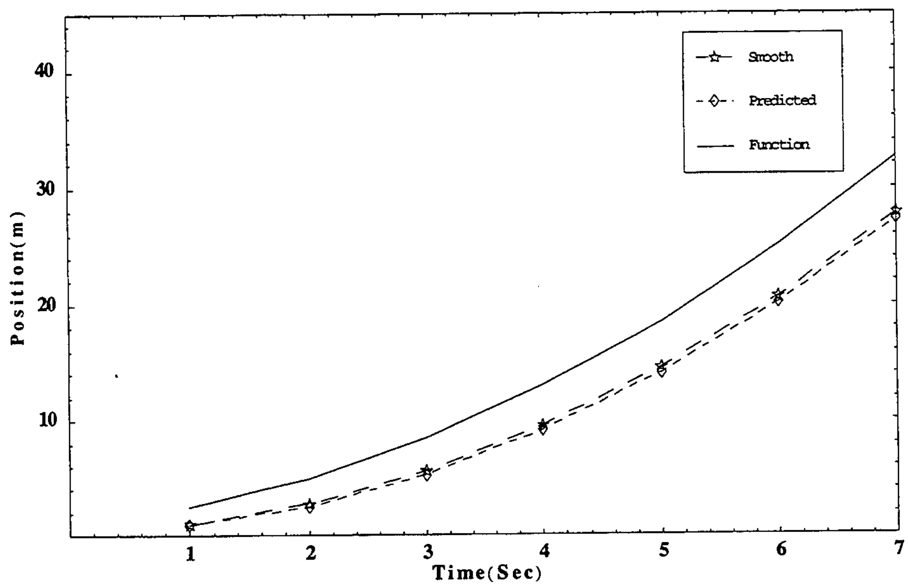

Figure 4.1-3. Constant Acceleration Motion Model with $\alpha = .1$

4.2 JERK MOTION MODEL

If one assumes that $x_m = \frac{y_0 m^3 T^3}{3!}$, then the general solution for the Jerk Motion Model becomes

$$\begin{aligned} n_k &= \xi\{\{\left(qt^{k+1}\sin\left((k-2)\varphi\right) - q^2t^k\sin\left((k-1)\varphi\right) + q^kt^2\sin(\varphi)\right) I \qquad (4\text{-}68) \\ &\quad + \left(-t^{k+1}\sin\left((k-2)\varphi\right) + qt^{k-1}\sin\left(\varphi\right) + q^kt\sin(2\varphi)\right) F_\gamma \\ &\quad + \left(t^k\sin\left((k-1)\varphi\right) - qt^{k-1}\sin\left(k\varphi\right) + q^k\sin(\varphi)\right) F_\gamma^2\}\eta_0 \\ &\quad + \sum_{m=0}^{k-1}\left[\left((Bt^m\sin(m\varphi) + (Dt^m\cos(m\varphi)) + E\right)\left(\frac{y_0m^3T^3}{3!}\right)\right] G_\beta. \end{aligned}$$

Consider only the inhomogeneous solution of Eq. (4-68),

$$\eta_k^i = \xi\sum_{m=0}^{k-1}\left[\left((Bt^m\sin(m\varphi) + (Dt^m\cos(m\varphi)) + E\right)\left(\frac{y_0m^3T^3}{3!}\right)\right] G_\beta \qquad (4\text{-}69)$$

$$= \frac{\xi y_0 T^3}{6} \sum_{m=0}^{k-1} \left((Bm^3 t^m \sin(m\varphi) + \left(Dm^3 t^m \cos(m\varphi)\right) + Em^3 \right) G_\beta,$$

$$= \frac{\xi y_0 T^3}{6} \left((BS_3(t,\varphi) + (DC_3(t,\varphi)) + \sum_{m=0}^{k-1} Em^3 \right) G_\beta$$

where

$$S_3(t,\varphi) = \sum_{m=0}^{k-1} m^3 t^m \sin(m\varphi), \tag{4-70}$$

$$C_3(t,\varphi) = \sum_{m=0}^{k-1} m^3 t^m \cos(m\varphi), \tag{4-71}$$

$$B = -q(q\cos[\varphi] - t\cos[2\varphi])I - t\cos[2\varphi]F_\gamma + t(-q + t\cos[\varphi])F_\gamma^2, \tag{4-72}$$

$$D = \sin[\varphi]\{q(q - 2t\cos[\varphi])I + 2t\cos[\varphi]F_\gamma - F_\gamma^2\}, \tag{4-73}$$

and

$$E = \sin[\varphi]\{q^m t^2 I + qt^{m-1}(-1 + 2qt\cos[\varphi])F_\gamma + q^m F_\gamma^2\}. \tag{4-74}$$

By evaluating $S_3(t,\varphi)$ and $C_3(t,\varphi)$, as done in Appendix A, and considering only the terms independent of m. The terms independent of m in the inhomogeneous solution simplify to

$$\begin{bmatrix} \frac{(1-\alpha)T^3}{\gamma} \\ \frac{(\alpha - \frac{\beta}{2} + \frac{\gamma}{12})T^2}{\gamma} \\ \left(\frac{\beta}{\gamma} + \frac{1}{2}\right)T \end{bmatrix} \tag{4-75}$$

Recall

$$n_k^i = \begin{bmatrix} x_s \\ v_s \\ a_s \end{bmatrix}, \tag{4-76}$$

which implies

$$l_p = \frac{(1-\alpha)\,T^3}{\gamma} \tag{4-77}$$

where l_p is the position lag,

$$l_v = \frac{\left(\alpha - \frac{\beta}{2} + \frac{\gamma}{12}\right)T^2}{\gamma} \tag{4-78}$$

where l_v is the velocity lag, and

$$l_a = \left(\frac{\beta}{\gamma} + \frac{1}{2}\right)T \tag{4-79}$$

where l_a is the acceleration lag.

The figures listed below of the Jerk Motion Model illustrate the convergence properties of the filter. Note that for α close to one, the filter converges quickly to steady state. While for smaller α, the filter converges very slowly to steady state.

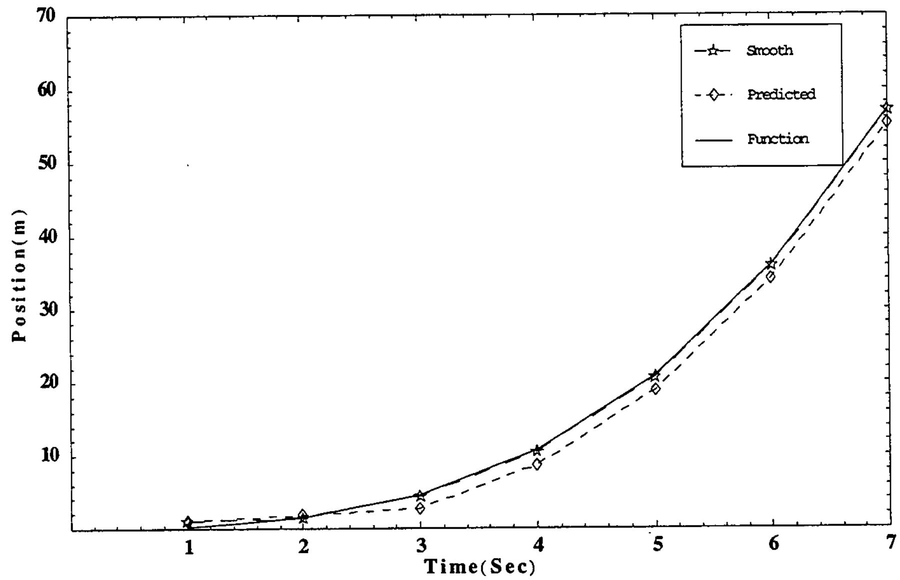

Figure 4.2-1. Jerk Motion Model with $\alpha = .9$

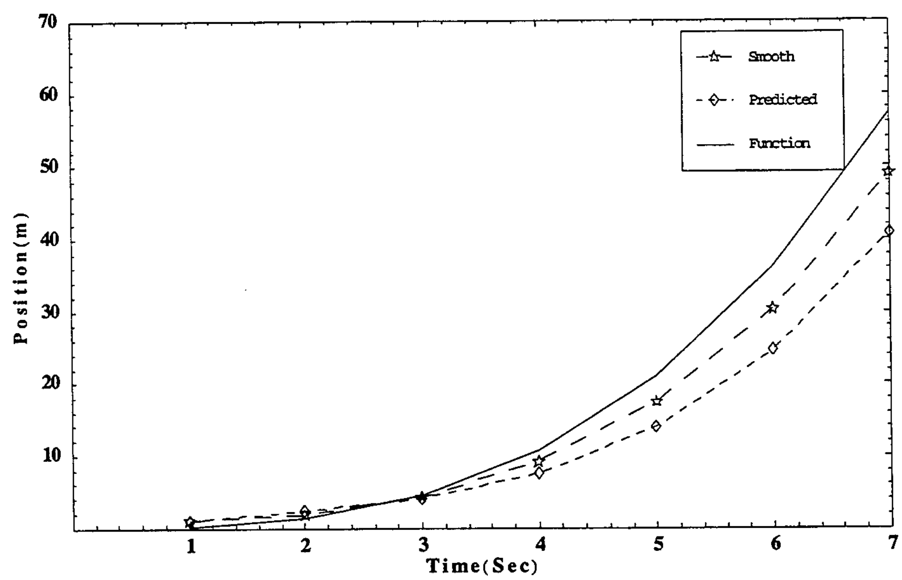

Figure 4.2-2. Jerk Motion Model with $\alpha = .5$

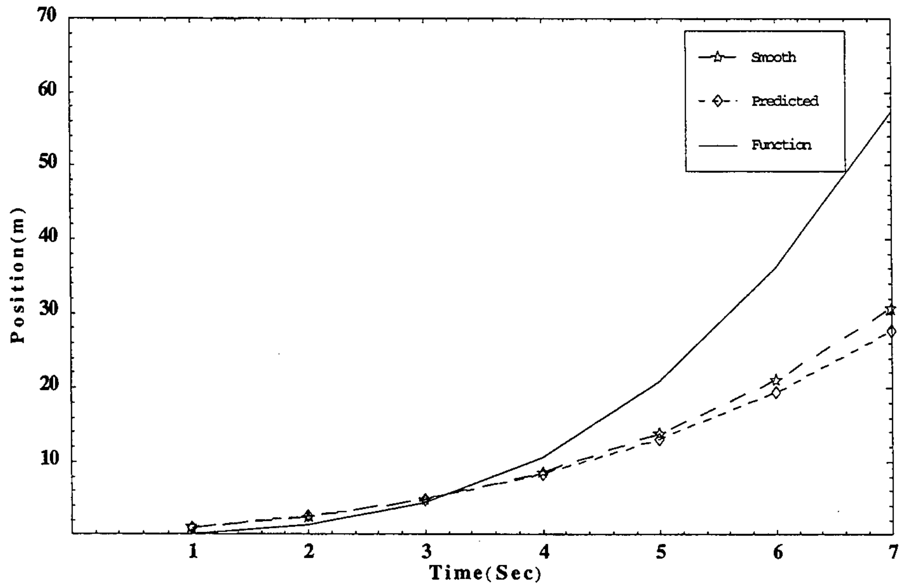

Figure 4.2-3. Jerk Motion Model with $\alpha = .1$

4.3 SINUSOIDAL MOTION MODEL

If one assumes that $x_m = \sin(mT)$, then the general solution for the Sinusoidal Motion Model becomes

$$\begin{aligned} n_k = {} & \xi\{\{\left(qt^{k+1}\sin((k-2)\varphi) - q^2t^k\sin((k-1)\varphi) + q^kt^2\sin(\varphi)\right) I \\ & + \left(-t^{k+1}\sin((k-2)\varphi) + qt^{k-1}\sin(\varphi) + q^kt\sin(2\varphi)\right) F_\gamma \\ & + \left(t^k\sin((k-1)\varphi) - qt^{k-1}\sin(k\varphi) + q^k\sin(\varphi)\right) F_\gamma^2\}\eta_0 \\ & + \sum_{m=0}^{k-1} [((Bt^m\sin(m\varphi) + (Dt^m\cos(m\varphi)) + E)(\sin(mT))] G_\beta. \end{aligned} \tag{4-80}$$

The figures listed below of the Sinusoidal Motion Model illustrate the convergence properties of the filter. Note that for α close to one, the filter converges quickly to steady state. While for smaller α, the filter converges very slowly to steady state.

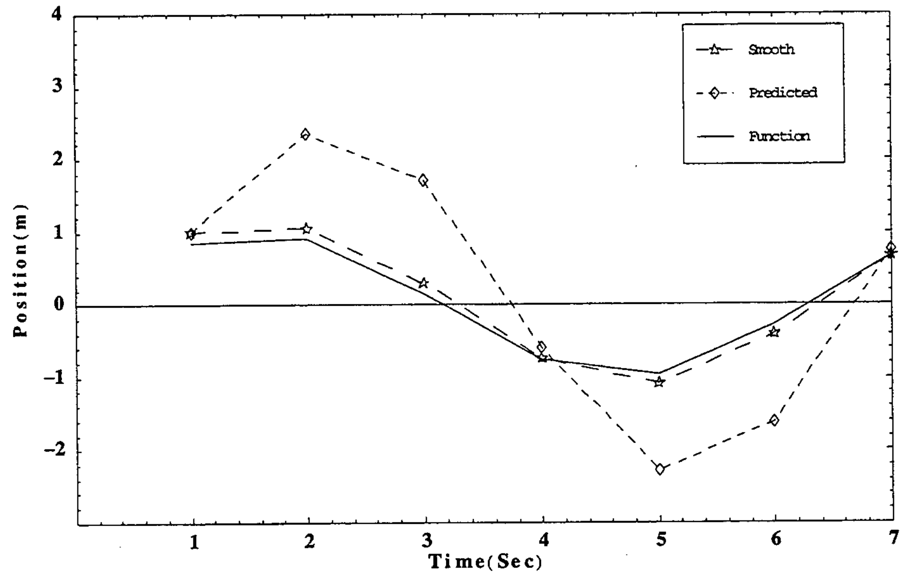

Figure 4.3-1. Sinusoidal Motion Model with $\alpha = .9$

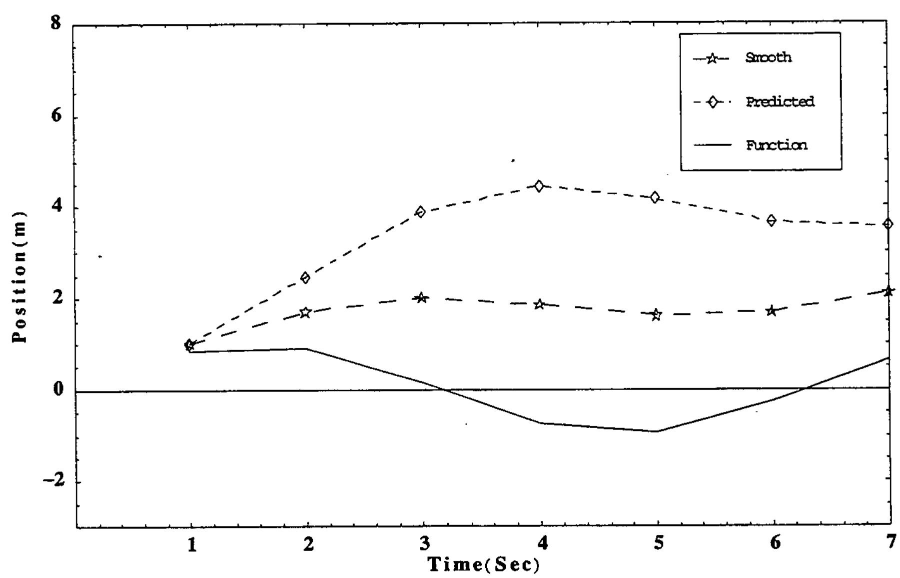

Figure 4.3-2. Sinusoidal Motion Model with $\alpha = .5$

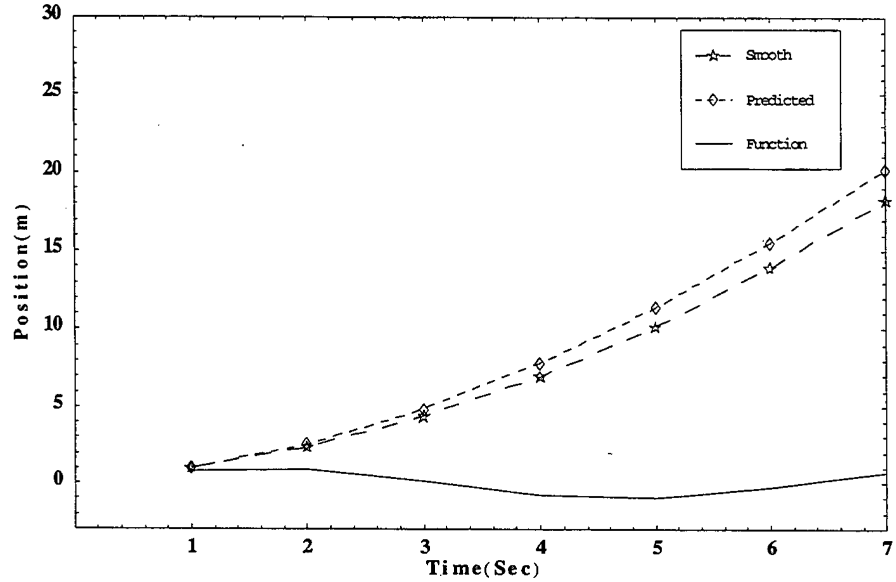

Figure 4.3-3. Sinusoidal Motion Model with $\alpha = .1$

5 NOISE REDUCTION RATIOS

Recall that one can express the update equations for a constant gain filter in the following form

$$\eta_{k+1} = F\eta_k + x_{k+1}G, \tag{5-1}$$

where $E(x_n) = 0$ and independent of η_k. The covariance matrix of a first order system is defined as [15] (where t denotes transpose)

$$\begin{aligned} P_k &= E\{(\eta_k - E(\eta_k))(\eta_k - E(\eta_k))^t\} \\ &= E\{(n_k - \bar{n}_k)(n_k - \bar{n}_k)^t, \end{aligned} \tag{5-2}$$

where $E(\eta_k) = \bar{\eta}_k = F\bar{\eta}_{k-1}$. One can calculate P_k by substituting Eq. (5-1) into Eq. (5-2)

$$\begin{aligned} P_k &= E\{(F\eta_{k-1} + x_kG - F\bar{\eta}_{k-1})\,(F\eta_{k-1} + x_kG - F\bar{\eta}_{k-1})^t\} \\ &= E\{F\eta_{k-1}\eta_{k-1}^tF^t + x_k^2GG^t\}. \end{aligned} \tag{5-3}$$

The expected value is explicitly given as

$$\begin{aligned} P_k &= \sum_{k=0}^{\infty}\{F\eta_{k-1}\eta_{k-1}^tF^t + x_{k+1}^2GG^t\} \\ &= F\left(\sum_{k=0}^{\infty}\eta_{k-1}\eta_{k-1}^t\right)F^t + \left(\sum_{k=0}^{\infty}x_{k+1}^2\right)GG^t \\ &= FP_{k-1}F^t + \sigma_n^2GG^t \end{aligned} \tag{5-4}$$

where $E(x_nx_n) = \sigma_n^2$. In steady state ($P_k = P_{k-1} = P$), so the covariance is in the form of a Lyapunov matrix equation

$$FPF^t - P = -\sigma_n^2GG^t, \tag{5-5}$$

Define a matrix S such that

$$PF^t - FP = S \tag{5-6}$$

By adding Eq. (5-5) and Eq. (5-6), one can solve for P explicitly in terms of known quantities to give

$$FPF^t - P + PF^t - FP = S - \sigma_n^2GG^t \tag{5-7}$$

$$(F+I)P(F^t - I) = S - \sigma_n^2GG^t \tag{5-8}$$

$$P = (F+I)^{-1}(S - \sigma_n^2GG^t)(F^t - I)^{-1} \tag{5-9}$$

We now have an expression of P. To solve for S, we pre-multiply Eq. (5-6) by F and postmultiply Eq. (5-6) by F^t which gives

$$FP\left(F^t\right)^2 - (F)^2 PF^t = FSF^t \tag{5-10}$$

then substituting from Eq. (5-5)

$$FPF^t = P - \sigma_n^2 GG^t \tag{5-11}$$

into Eq. (5-10) gives the solution for S

$$FSF^t - S = F\sigma_n^2 GG^t - \sigma_n^2 GG^t F^t \tag{5-12}$$

One can then solve this equation for S and then substitute S into Eq. (5-9) to get the value of P.

5.1 NOISE REDUCTION RATIOS FOR THE $\alpha - \beta$ FILTER

For the $\alpha - \beta$ filter, recall

$$F_\beta = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\frac{\beta}{T} & 1-\beta \end{bmatrix} \tag{5-13}$$

and

$$G_\beta = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \end{bmatrix}. \tag{5-14}$$

By substituting F_β and G_β into Eq. (5-12)

$$F_\beta S F_\beta^t - S = \left(F_\beta G_\beta G_\beta^t - G_\beta G_\beta^t F_\beta^t\right)\sigma_n^2 \tag{5-15}$$

$$F_\beta S F_\beta^t - S = \sigma_n^2 \begin{bmatrix} 0 & \beta^2 \\ -\beta^2 & 0 \end{bmatrix} \tag{5-16}$$

Solving for S, yields

$$S = \frac{1}{f_\beta}\begin{bmatrix} 0 & \beta^3 \\ -\beta^3 & 0 \end{bmatrix} \tag{5-17}$$

where

$$f_\beta = -\alpha\beta \tag{5-18}$$

One can now solve for P by substituting S into Eq. (5-9)

$$\begin{aligned} P &= (F_\beta + I)^{-1}(S - \sigma_n^2 G_\beta G_\beta^t)(F_\beta^t - I)^{-1} \qquad (5\text{-}19) \\ &= \begin{bmatrix} \frac{2-\beta}{4-2\alpha-\beta} & \frac{-T(1-\alpha)}{4-2\alpha-\beta} \\ \frac{\beta}{T(4-2\alpha-\beta)} & \frac{2-\alpha}{4-2\alpha-\beta} \end{bmatrix} \begin{bmatrix} -\alpha^2 & -\frac{\beta}{T}(\alpha + \frac{\beta}{\alpha}) \\ -\frac{\beta}{T}(\alpha - \frac{\beta}{\alpha}) & -\frac{\beta^2}{T} \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{T} \\ \frac{-T(1-\alpha)}{\beta} & \frac{-\alpha}{\beta} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{-2\alpha^2+\beta(3\alpha-2)}{\alpha(-4+2\alpha+\beta)} & \frac{-\beta(2\alpha-\beta)}{\alpha T(-4+2\alpha+\beta)} \\ \frac{-\beta(2\alpha-\beta)}{\alpha T(-4+2\alpha+\beta)} & \frac{-2\beta^2}{\alpha T^2(-4+2\alpha+\beta)} \end{bmatrix}$$

The elements of this covariance matrix reveal the noise reduction ratios for position (P_x), velocity (P_v), and position x velocity(P_{xv}).

$$P_x(0) = \frac{-2\alpha^2 + \beta(3\alpha - 2)}{\alpha(-4 + 2\alpha + \beta)}, \tag{5-20}$$

$$P_v(0) = \frac{-2\beta^2}{\alpha T^2(-4 + 2\alpha + \beta)}, \tag{5-21}$$

and

$$P_{xv}(0) = \frac{-\beta(2\alpha - \beta)}{\alpha T(-4 + 2\alpha + \beta)}. \tag{5-22}$$

5.2 NOISE REDUCTION RATIOS FOR THE $\alpha - \beta - \gamma$ FILTER

For the $\alpha - \beta - \gamma$ filter, recall

$$F_\gamma = \begin{bmatrix} 1-\alpha & (1-\alpha)T & (1-\alpha)\frac{T^2}{2} \\ -\frac{\beta}{T} & 1-\beta & \left(1-\frac{\beta}{2}\right)T \\ -\frac{\gamma}{T^2} & -\frac{\gamma}{T} & 1-\frac{\gamma}{2} \end{bmatrix} \tag{5-23}$$

and

$$G_\gamma = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \\ \frac{\gamma}{T^2} \end{bmatrix}. \tag{5-24}$$

By substituting F_γ and G_γ into Eq. (5-12)

$$F_\gamma S F_\gamma^t - S = \left(F_\gamma G_\gamma G_\gamma^t - G_\gamma G_\gamma^t F_\gamma^t\right)\sigma_n^2 \tag{5-25}$$

$$F_\gamma S F_\gamma^t - S = \sigma_n^2 \begin{bmatrix} 0 & \frac{2\beta^2-2\alpha\beta+\beta\gamma}{2T} & \frac{\gamma(2\beta+\gamma)}{2T^2} \\ \frac{-2\beta^2+2\alpha\beta-\beta\gamma}{2T} & 0 & \frac{\gamma^2}{T^3} \\ \frac{-\gamma(2\beta+\gamma)}{2T^2} & -\frac{\gamma^2}{T^3} & 0 \end{bmatrix} \tag{5-26}$$

Solving for S, yields

$$S = \frac{1}{f_\gamma}\begin{bmatrix} 0 & \beta^3 - \frac{1}{4}\beta\gamma^2 & \beta^2\gamma - \frac{\gamma^3}{4} \\ \frac{1}{4}\beta\gamma^2 - \beta^3 & 0 & \gamma^2\beta + \frac{\gamma^3}{2} \\ \frac{\gamma^3}{4} - \beta^2\gamma & -\gamma^2\beta - \frac{\gamma^3}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5-27}$$

such that

$$f_\gamma = -\gamma - \frac{1}{2}\alpha\gamma - \alpha\beta \tag{5-28}$$

One can solve for P by substituting S into Eq. (5-9)

$$\begin{aligned} P &= (F_\gamma + I)^{-1}(S - \sigma_n^2 G_\gamma G_\gamma^t)(F_\gamma^t - I)^{-1} \qquad (5\text{-}29) \\ &= \frac{\sigma_n^2}{h(4-2\alpha-\beta)} \begin{bmatrix} 2\alpha h - \beta^2(6\alpha-4) + \alpha\beta\gamma & \frac{\beta(2\alpha-\beta)(2\beta-\gamma)}{T} & \frac{(2\gamma h+\beta\gamma(\gamma-2\beta))}{T^2} \\ \frac{\beta(2\alpha-\beta)(2\beta-\gamma)}{T} & \frac{(2\gamma^2(2-\alpha)+4\beta^2(\beta-\gamma))}{T^2} & \frac{2\beta\gamma(2\beta-\gamma)}{T^3} \\ \frac{2\gamma h+\beta\gamma(\gamma-2\beta)}{T^2} & \frac{2\beta\gamma(2\beta-\gamma)}{T^3} & \frac{4\beta\gamma^2}{T^4} \end{bmatrix} \end{aligned}$$

where

$$h = 2\alpha\beta + \alpha\gamma - 2\gamma \tag{5-30}$$

since

$$(F_\gamma + I)^{-1} = \begin{bmatrix} \frac{4-2\beta}{8-4\alpha-2\beta} & -2T(1-\alpha) & 0 \\ \frac{\frac{2\beta}{T}-\frac{\gamma}{T}}{8-4\alpha-2\beta} & \frac{4-2\alpha-\frac{\gamma}{2}}{8-4\alpha-2\beta} & \frac{-2T+T\alpha+\frac{T\beta}{2}}{8-4\alpha-2\beta} \\ \frac{2\gamma}{T^2(8-4\alpha-2\beta)} & \frac{\gamma}{T(8-4\alpha-2\beta)} & \frac{4-2\alpha-\beta}{8-4\alpha-2\beta} \end{bmatrix}, \tag{5-31}$$

$$S - \sigma_n^2 G_\gamma G_\gamma^t = \begin{bmatrix} -\alpha^2 & -\frac{\alpha\beta}{T} + \frac{g+2\alpha\gamma^2+\beta\gamma^2}{2h} & \frac{-\alpha\gamma}{T^2} - \frac{(2\alpha-\beta)\gamma(2\beta+\gamma)}{2h} \\ -\frac{\alpha\beta}{T} - \frac{g+2\alpha\gamma^2+\beta\gamma^2}{2h} & -\frac{\beta^2}{T^2} & \frac{-\beta\gamma}{T^3} - \frac{\gamma^2(2\beta+\gamma)}{h} \\ -\frac{\alpha\gamma}{T} + \frac{(2\beta-\gamma)\gamma(2\beta+\gamma)}{2h} & \frac{-\beta\gamma}{T^3} + \frac{\gamma^2(2\beta+\gamma)}{h} & -\frac{\gamma^2}{T^4} \end{bmatrix}, \tag{5-32}$$

where

$$g = -4\beta^3 + 4\alpha\beta\gamma - 4\gamma^2, \tag{5-33}$$

and

$$h = 2\alpha\beta + \alpha\gamma - 2\gamma \tag{5-34}$$

and

$$(F_\gamma^t - I)^{-1} = \begin{bmatrix} -1 & \frac{1}{T} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{T} \\ -\frac{T^2-T2\alpha}{\gamma} & -\frac{T\alpha-\frac{T\beta}{2}}{\gamma} & \frac{-\beta}{\gamma} \end{bmatrix}. \tag{5-35}$$

The elements of this covariance matrix reveal the noise reduction ratios for position (P_x), velocity (P_v), acceleration (P_a), position x velocity (P_{xv}), and velocity x acceleration (P_{va})

$$P_x(0) = \frac{\left(2\alpha(2\alpha\beta+\alpha\gamma-2\gamma) - \beta^2(6\alpha-4) + \alpha\beta\gamma\right)\sigma_n^2}{(2\alpha\beta+\alpha\gamma-2\gamma)(4-2\alpha-\beta)}, \tag{5-36}$$

$$P_v(0) = \frac{-(2\gamma^2(2-\alpha) + 4\beta^2(\beta-\gamma))\sigma_n^2}{T^2(2\alpha\beta+\alpha\gamma-2\gamma)(4-2\alpha-\beta)}, \tag{5-37}$$

$$P_{xv}(0) = \frac{\left(2\alpha(2\alpha\beta+\alpha\gamma-2\gamma) - \beta^2(6\alpha-4) + \alpha\beta\gamma\right)\sigma_n^2}{(2\alpha\beta+\alpha\gamma-2\gamma)(4-2\alpha-\beta)}, \tag{5-38}$$

$$P_a(0) = \frac{4\beta\gamma^2\sigma_n^2}{T^4(2\alpha\beta + \alpha\gamma - 2\gamma)(4 - 2\alpha - \beta)}, \tag{5-39}$$

and

$$P_{va}(0) = \frac{2\beta\gamma(2\beta - \gamma)\sigma_n^2}{T^3(2\alpha\beta + \alpha\gamma - 2\gamma)(4 - 2\alpha - \beta)}. \tag{5-40}$$

6 COST FUNCTIONS AS ALTERNATIVE TO TRACKING INDEX

Independent of which of the three relationships between α and β one assumes, each relationship can be shown to obey the common constraint due to Kalata [13]

$$\Gamma^2 = \frac{\beta^2}{1-\alpha} \tag{6-1}$$

where the variable Γ is commonly known in the naval community as the Kalata tracking index,

$$\Gamma^2 = \frac{T^4\sigma_a^2}{\sigma_m^2}. \tag{6-2}$$

The tracking index is a function of the assumed target maneuverability variance σ_a^2 (deviation from modeled behavior), radar measurement noise variance σ_m^2, and T is the update interval. The maneuverability is an unknown parameter in most cases because there is no direct means of determining it from system parameters, nor is it measurable. One invents a process noise model that is considered the best means of modeling unknown threat behavior. The variance in maneuverability is σ_a^2.

There are alternative means of selecting the values for the filter coefficients than the tracking index. This method is based on the fact that any unmodeled error in a filter introduces a bias which is characteristic of the filter response. A cost function that combines noise reduction and bias is introduced and minimized with respect to the α if used in the filter equations (one has, of course, reduced β to $\beta = \beta(\alpha)$). This methodology amounts to accepting a mean squared error as the arbitrator of performance characterization. One could adapt other criteria as well [24], but mean square is widely accepted.

The response of an $\alpha - \beta$ filter to a linear acceleration produces three terms; a transient term, a lag, and the model input term. The lag is a bias, so the expected value E of the steady state response $E[x_s(k) - x_m(k)]$ is the lag. Similarly,

$$E[(x_s(k) - x_m(k))^2] = P_x(0)\sigma_n^2 + \frac{1}{2}L_x^2 a_0^2 T^4 \tag{6-3}$$

and

$$E[(v_s(k) - x_m(k))^2] = P_v(0)\sigma_n^2 + L_v^2 a_0^2 T^2. \tag{6-4}$$

This allows one to define a specific cost function [23] for the smoothed velocity

$$\begin{aligned} J_V(\alpha,\beta) &= E[(v_s(k) - x_m(k))^2] \\ &= \sigma_n^2 P_v(0) + \left(\frac{\alpha}{\beta} - \frac{1}{2}\right)^2 a_0^2 T^2. \end{aligned} \tag{6-5}$$

where the terms have been previously defined. By introducing a change of variable

$$\tau = \frac{\alpha}{\beta}, \tag{6-6}$$

one can cast the second term of the cost function that is invariant with respect to the relationship chosen between α and β. The velocity noise reduction ratio can be written as

$$\begin{aligned} P_v(0) &= \frac{-2\beta^2}{\alpha T^2(-4+2\alpha+\beta)} \\ &= \frac{-2}{\frac{\tau}{\beta}T^2(-4+2\alpha+\beta)} \\ &= \frac{2\beta}{\tau T^2(4-2\beta\tau-\beta)} \end{aligned} \tag{6-7}$$

which can be cast purely in terms of τ once the relationship between α and β is made. Therefore, the specific cost function for smooth velocity is of the form

$$\begin{aligned} J_V(\alpha,\beta) &= \sigma_n^2 P_v(0) + \left(\tau - \frac{1}{2}\right)^2 a_0^2 T^2 \\ &= \sigma_n^2\left(\frac{2\beta}{\tau T^2(4-2\beta\tau-\beta)}\right) + \left(\tau - \frac{1}{2}\right)^2 a_0^2 T^2 \end{aligned} \tag{6-8}$$

For the Benedict-Bordner relation $\beta_{BB} = \frac{\alpha^2}{2-\alpha}$, substituted into $\tau = \frac{\alpha}{\beta}$ gives

$$\alpha^B = \frac{2}{(\tau+1)} \tag{6-9}$$

and

$$\beta^B = \frac{2}{\tau(\tau+1)}. \tag{6-10}$$

When one uses the Benedict-Bordner filter relation, one finds the velocity noise reduction ratio

$$\begin{aligned} P_v^B &= \frac{2\beta}{\tau T^2(4-2\beta\tau-\beta)} \\ &= \frac{2}{\tau T^2(2\tau^2-1)} \end{aligned} \tag{6-11}$$

For the Kalata relation $\beta = 2(2-\alpha) - 4\sqrt{1-\alpha}$, substituted into $\tau = \frac{\alpha}{\beta}$ gives

$$\alpha^K = \frac{8\tau}{(2\tau+1)^2} \tag{6-12}$$

and

$$\beta^K = \frac{8}{(2\tau+1)^2}. \tag{6-13}$$

When one uses the Kalata filter relation, one finds the velocity noise reduction ratio

$$\begin{aligned} P_v^K &= \frac{2\beta}{\tau T^2(4-2\beta\tau-\beta)} \\ &= \frac{4}{\tau T^2(4\tau^2-1)} \end{aligned} \tag{6-14}$$

For the CTWN relation $\alpha = \sqrt{2\beta + \frac{\beta^2}{12}} - \frac{\beta}{2}$, substituted into $\tau = \frac{\alpha}{\beta}$ gives

$$\alpha^C = \frac{12\tau}{(6\tau^2+6\tau+1)} \tag{6-15}$$

and

$$\beta^C = \frac{12}{(6\tau^2+6\tau+1)}. \tag{6-16}$$

When one uses the CTWN filter relation, one finds the velocity noise reduction ratio

$$\begin{aligned} P_v^C &= \frac{2\beta}{\tau T^2(4-2\beta\tau-\beta)} \\ &= \frac{3}{\tau T^2\left(3\tau^2-1\right)} \end{aligned} \tag{6-17}$$

In general, when one uses the different filter relationships, one finds the velocity noise reduction ratio (T=1)

$$P_V^\alpha = \frac{d}{\tau\left(d\tau^2-1\right)} \tag{6-18}$$

where $d = 2$ for Benedict-Bordner, $d = 4$ for Kalata, and $d = 3$, for CWTN. Eq. (6-8) can now be written as a normalized cost function (e.g. $\frac{T^2 J_V}{\sigma_n^2} = J$)

$$J^d\left(\tau, \Lambda_R\right) = \frac{d}{\tau\left(d\tau^2-1\right)} + \Gamma_R^2\left(\tau - \frac{1}{2}\right)^2 \tag{6-19}$$

where $\Gamma_R = \frac{a_0 T^2}{\sigma_n}$. Note that Γ_R has been rewritten with a subscript to avoid confusion with the Kalata tracking index which has the same physical units but a different interpretation. There are a couple of interesting things to note about the cost function. First, the bias has been reduced to a form that is invariant with respect to the relationship between the filter coefficients. This means that one can minimize the cost function to arrive at $\tau = \tau(\Gamma)$, hence one can arrive at a selection criteria that has an absolute invariance with respect to lag. Additionally, one can maintain the measure of performance; namely that the filter coefficient are minimized jointly between velocity noise reduction and "velocity lag". This leads to different performance than the Kalata criteria for selecting the filter coefficients which is based on a plant noise model as a means of selecting the filter coefficients.

To compare the performance of different coefficient selection techniques, one can first show how to compute the coefficients. Taking the derivative of Eq. (6-19) with respect to τ and setting it equal to zero gives

$$\Gamma_R^2 = \frac{d\,(3d\tau^2 - 1)}{\tau^2(d\tau^2 - 1)^2\,(2\tau - 1)}. \tag{6-20}$$

This can be solved numerically or graphically to give $\tau = \tau\,(\Gamma_R)$ which solves the filter selection coefficient problem. Substituting Eq. (6-20) into Eq. (6-19) gives

$$J^a(\tau_{optimized}) = \frac{d}{\tau\,(d\tau^2 - 1)} + \frac{d(3d\tau^2 - 1)\left(\tau - \frac{1}{2}\right)}{2\tau^2\,(d\tau^2 - 1)^2}, \tag{6-21}$$

which is cost function evaluated at the optimized value of τ. The tracking index is

$$\Gamma^2 = \frac{4}{\tau^2\,(\tau^2 - 1)} \tag{6-22}$$

for the Benedict-Bordner relationship

$$\Gamma^2 = \frac{64}{16\tau^4 - 8\tau^2 - 1} \tag{6-23}$$

for the Kalata relationship,

$$\Gamma^2 = \frac{144}{36\tau^4 - 24\tau^2 + 1} \tag{6-24}$$

for the CTWN relationship. To make a comparison, one would substitute each one of these equations into Eq. (6-19) and compare the plot for all τ.

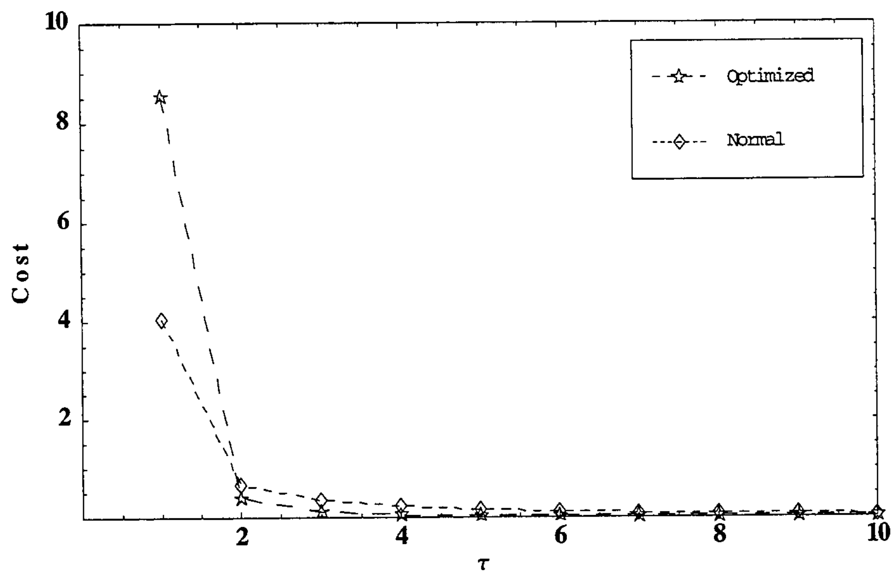

Figure 6.1-1. Benedict-Bordner Relation

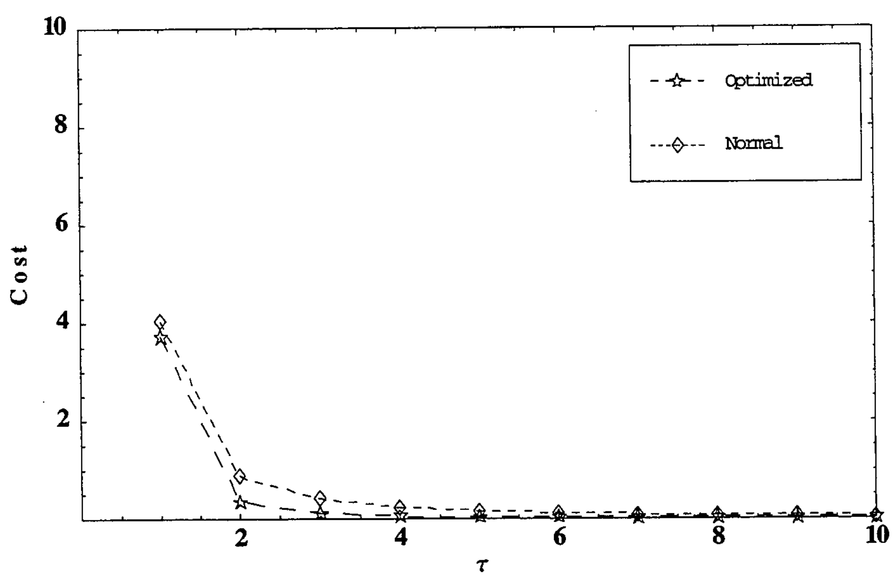

Figure 6.1-2. Kalata Relation

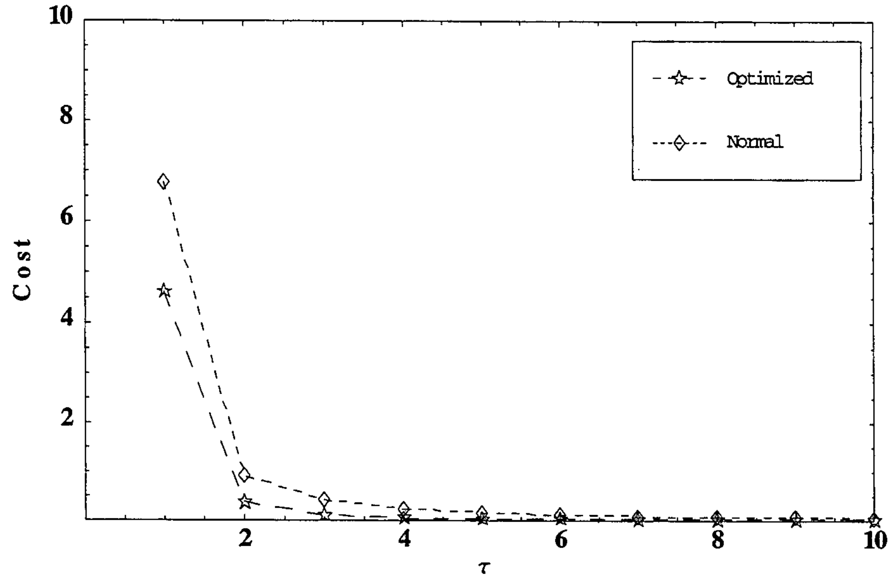

Figure 6.1-3. CTWN Relation

The relationship between α and τ is expressed in the following figure.

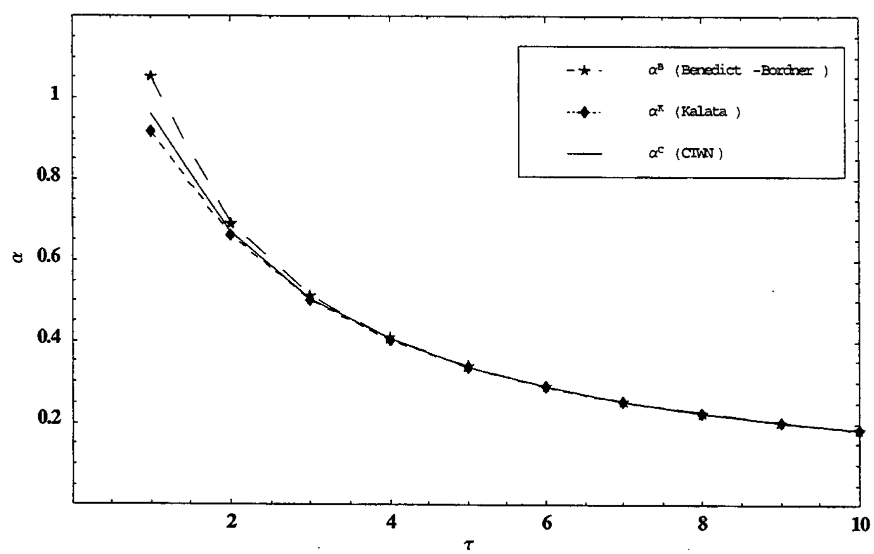

Figure 6.1-4. Relation between α and τ.

For a given τ, one sees that the Kalata relation gives the smallest α for a given τ.

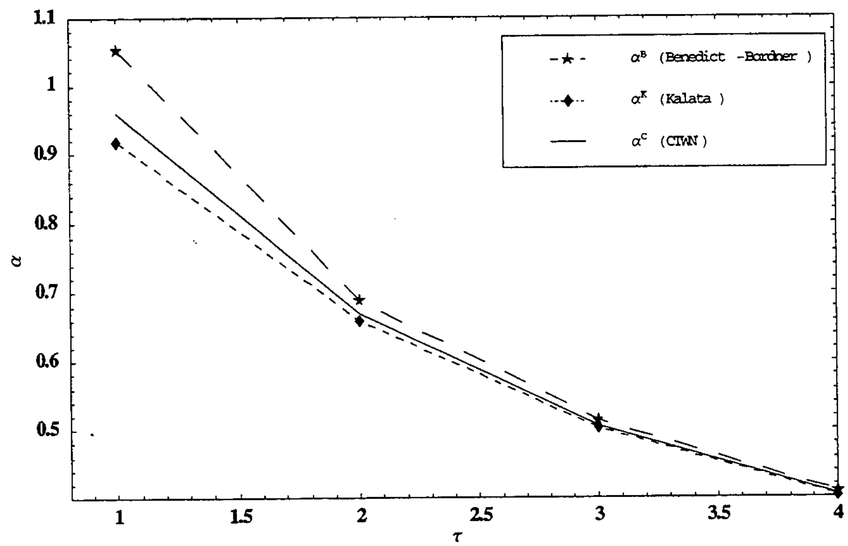

Figure 6.1-5. Isolated Comparison Between α and τ

The result of this comparison is that one will have smaller cost for the optimization technique, which is indicative of better performance. Note, this performance improvement occurs when there is unmodeled behavior only, so this result is useful in a complex target environment, but not necessarily in a benign threat environment.

Different cost functions can be used for problems other than the ones already discussed might prove useful under some circumstances. If there is concern about transient effects that occur while the target is accelerating, replacing the lags by twice their value includes this effect since the transient response is always less than or equal to the lag. Another possibility is to weight the lag by the percentage of time maneuvers are expected to occur. In some cases, it is more desirable to minimize the lag in for the predicted filter response rather than the acceleration lag. If minimum mean square in position is the primary design criteria (say as might be used in track maintenance), then applying the same analysis for the position cost function is appropriate. Furthermore, exactly the same type of cost function approach can be applied to the $\alpha - \beta - \gamma$ filter. Further discussion of the details of this will be presented in a future report which will concentrate on combining these results with a track maintenance implementation of a multiple model $\alpha - \beta/\alpha - \beta - \gamma$ filter that will be used to illustrate how to bound IMM performance in a manner useful to Naval applications. The next section illustrates how the cost-function technique can be used in an implementation of a table-driven filter implementation.

6.1 DESIGN EXAMPLES

To illustrate that, with proper initialization of the radar track data, the $\alpha - \beta$ filter can be implemented as a look-up table of the coefficients as a function of range achieves similar performance as a Kalman filter. Unless one is tracking only a few objects, the $\alpha - \beta$ filter implementation of the Kalman filter could be preferred over the conventional Kalman filter for real-time AAW applications. It does require some care in the design in terms of understanding the operating environment as well as careful thought in the underlying design concepts. With that understanding, one can consider several specific examples. When tracking is done in Cartesian coordinates, the noise is range dependent. Typically, the noise can be written as $\sigma_x = R\sigma_\theta$, where σ_θ is the sensor angular noise, which is a known parameter of the tracking system. Given that the range is in kilometers and the angle noise is in milliradians, the noise can be written as a function of the range (n) times a constant k. The tracking index

$$\Gamma = \frac{a_0 T^2}{kn}. \tag{6-25}$$

Specific system parameters are then plugged into Eq. (6-25) and the $\alpha's$ are computed. For example, commercial aircrafts do not have maneuvers that exceed $2g's$. For simplicity, choose other system parameters so that the tracking index is $\Gamma = \frac{1.76}{n}$. Table 6.1.1 shows the two different $\alpha's$ computed from the common tracking index for ranges from $4 - 128\ km$.

$Range(n)$	Γ	α_K	$P_x^K(0)$	$P_v^K(0)$	$J_v^K(\tau,\Gamma)$	α_R	$P_x^R(0)$	$P_v^R(0)$	$J_v^R(\tau,\Gamma)$
4	.44	.61	.52	.10	.65	.71	.46	.12	.38
8	.22	.48	.40	.04	.35	.61	.51	.10	.23
16	.11	.37	.30	.01	.19	.52	.38	.04	.10
32	.06	.28	.22	.005	.10	.43	.28	.01	.04
64	.03	.21	.16	.002	.05	.35	.20	.006	.019
128	.014	.15	.12	.001	.03	.28	.15	.002	.008

Table 6.1-1. Cost Function versus Range

Two other examples of interest to the tracking community can be mapped into the same form as the accelerating target with different interpretations of the coefficients used to form the tracking index. A maneuvering target that has a turning rate of less than 15^o deg/sec has an acceleration that is equivalent to a constant acceleration so the constant in the tracking index can be represented by $\rho_0\omega^2$ where ω is the turning rate and ρ_0 is the radius of curvature of the turn. The noise is now angular $\sigma_n \Rightarrow R\sigma_\theta$ and to a 98% confidence level, a becomes $3\sigma_\theta R/T$. Thus, this example is completely mapped into the solution in Eq. (6-25) with

$$\Gamma_m = \frac{\rho_0 \omega^2 T^2}{3R\sigma_\theta}. \tag{6-26}$$

Another example is to determine the ballistic coefficient of an object. The force on an object undergoing a ballistic slowdown is $F = \frac{1}{2}\rho k v^2$. This force produces an equivalent acceleration term $a_0 = \rho \frac{\left(\frac{k}{m}\right)^2}{v_0^2}$, which can be used to give the tracking index

$$\Gamma_m = \frac{\rho k^2 T^2}{3m^2 v_0^2 R \sigma_\theta}. \tag{6-27}$$

These examples illustrate the usefulness of this approach in that there is a great deal of flexibility to attack different problems, that can all be mapped into the same tracking index with a different definitions of its parameters.

7 DETERMINATION OF FILTER COEFFICIENTS

The selection of the relationship between α and β remains somewhat problematic in the discussion of filters. Because of the close connection between the steady state solution of the Kalman filter and the constant gain $\alpha - \beta$ filter, there has been no independent development of the coefficient relations other than the Benedict-Bordner relationship. This can be rectified as show below, but it remains largely an academic exercise for many. Thus it may be skipped by those who are uninterested without serious loss of continuity with the previous portions of the document.

Recall that cost functions in the previous chapter are used to determine α in terms of known system and design parameters. This cost function approach can also be used to determine filter coefficient relationships $\beta = \beta(\alpha)$. Recall from optimization theory [?], that for a function of two variables, say x and y, it is possible to minimize this function so that a minimum exists with respect to some objective criteria. Once the relevant equations are solved, a single functional relationship $y = y(x)$ is achieved that satisfies the minimization criteria. To illustrate how to do this, note that a cost function J is minimized by taking the gradient and taking the inner product with a vector normal to that function

$$\nabla J \cdot \hat{n} = 0. \tag{7-1}$$

To illustrate this, consider the fortuitous choice of a cost function

$$J(x,y) = a\frac{f(x,y)}{h(x,y)} + bg(x,y). \tag{7-2}$$

Taking the directional derivative of $J(x,y)$ with respect to x and y respectively, gives the matrix equation (with the normal vector corresponding to the matrix of parameters a and b)

$$\begin{bmatrix} \frac{(f_x h - f h_x)}{h^2(x,y)} & g_x \\ \frac{(f_y h - f h_y)}{h^2(x,y)} & g_y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{7-3}$$

Clearly this equation is satisfied only if the determinant is zero, which gives

$$g_y(f_x h - f h_x) - g_x(f_y h - f h_y) = 0. \tag{7-4}$$

Solving this then implies $y = y(x)$, which is equivalent to a filter coefficient relationship.

There are four obvious choices for the cost function. We have already discussed two. Recall that the noise reduction ratio for the smoothed velocity is

$$\begin{aligned} P_v(0) &= \frac{2\beta^2}{T^2\alpha(4 - 2\alpha - \beta)} \\ &= \frac{2\beta}{T^2\tau(4 - 2\tau\beta - \beta)} \end{aligned} \tag{7-5}$$

so a velocity-lag-noise-reduction cost function is

$$J_{NLv} = a\frac{\beta}{\tau(4-2\tau\beta-\beta)} + b(\tau - \frac{1}{2})^2. \tag{7-6}$$

Recall that the noise reduction ratio for the smoothed position is given by

$$\begin{aligned} P_x(0) &= \frac{2\alpha^2 + \beta(2-3\alpha)}{\alpha(4-2\alpha-\beta)} \\ &= \frac{2\beta\tau^2 + (2-3\tau\beta)}{\tau(4-2\tau\beta-\beta)} \end{aligned} \tag{7-7}$$

so a position-lag-noise-reduction cost function is

$$J_{NLp} = a\frac{2\beta\tau^2 + (2-3\tau\beta)}{\tau(4-2\tau\beta-\beta)} + b\frac{(1-\tau\beta)^2}{\beta^2}. \tag{7-8}$$

Two other choices for cost functions date back to Benedict [2]. The transient velocity performance of an $\alpha - \beta$ is

$$\begin{aligned} D_v &= \frac{1}{T^2}\frac{\alpha^2(2-\alpha) + 2\beta(1-\alpha)}{\alpha\beta(4-2\alpha-\beta)} \\ &= \frac{1}{T^2}\frac{\beta\tau^2(2-\beta\tau) + 2(1-\beta\tau)}{\beta\tau(4-2\tau\beta-\beta)} \end{aligned} \tag{7-9}$$

so a transient-velocity-noise-reduction cost function is

$$J_{NTv} = a\frac{\beta}{\tau(4-2\tau\beta-\beta)} + b\frac{\beta\tau^2(2-\beta\tau) + 2(1-\beta\tau)}{\beta\tau(4-2\tau\beta-\beta)}. \tag{7-10}$$

The transient position performance of an $\alpha - \beta$ is

$$\begin{aligned} D_p &= \frac{(2-\alpha)(1-\alpha)^2}{\alpha\beta(4-2\alpha-\beta)} \\ &= \frac{(2-\beta\tau)(1-\beta\tau)^2}{\tau\beta^2(4-2\beta\tau-\beta)} \end{aligned} \tag{7-11}$$

so a transient-position-noise-reduction cost function is (which is the criteria used by Benedict to derive the BB relationship, though he used a somewhat convoluted z-transform technique)

$$J_{NTp} = a\frac{2\beta\tau^2 + (2-3\tau\beta)}{\tau(4-2\tau\beta-\beta)} + b\frac{(2-\beta\tau)(1-\beta\tau)^2}{\tau\beta^2(4-2\beta\tau-\beta)}. \tag{7-12}$$

7.1 VELOCITY-LAG-NOISE-REDUCTION COST FUNCTION

To determine which different coefficient relationship are achieved, first consider the velocity-lag-noise-reduction cost function, J_{NLv}. Note (from now on there is the mapping $x = \tau$ and $y = \beta$)

$$f(\tau, \beta) = \beta, \tag{7-13}$$

which implies

$$f_\tau(\tau, \beta) = 0, \tag{7-14}$$

and

$$f_\beta(\tau, \beta) = 1. \tag{7-15}$$

Also,

$$g(\tau, \beta) = \left(\tau - \frac{1}{2}\right)^2; \tag{7-16}$$

which implies

$$g_\tau(\tau, \beta) = (2\tau - 1), \tag{7-17}$$

and

$$g_\beta(\tau, \beta) = 0. \tag{7-18}$$

Finally, note

$$h(\tau, \beta) = \tau(4 - 2\tau\beta - \beta); \tag{7-19}$$

which implies

$$h_\tau(\tau, \beta) = (4 - 4\tau\beta - \beta), \tag{7-20}$$

and

$$h_\beta(\tau, \beta) = -(2\tau + 1)\tau \tag{7-21}$$

Additionally,

$$(f_\tau h - f h_\tau) = \beta(-4 + 4\tau\beta + \beta) \tag{7-22}$$

and

$$(f_\beta h - f h_\beta) = 4\tau \tag{7-23}$$

are needed. Substituting Eq. (7-17), Eq. (7-18), Eq. (7-22), and Eq. (7-23) into Eq. (7-4), gives

$$4\tau - 8\tau^2 = 0$$

which implies that there is no optimal relationship between the coefficients for this cost function.

7.2 POSITION-LAG-NOISE-REDUCTION COST FUNCTION

To determine which different coefficient relationship are achieved, first consider the position-lag-noise-reduction cost function, J_{NLp}; note that

$$f(\tau,\beta) = 2\beta\tau^2 + (2 - 3\tau\beta), \tag{7-24}$$

which implies

$$f_\tau(\tau,\beta) = 4\beta\tau - 3\beta, \tag{7-25}$$

and

$$f_\beta(\tau,\beta) = (2\tau^2 - 3\tau). \tag{7-26}$$

Also,

$$g(\tau,\beta) = \frac{(1-\beta\tau)^2}{\beta^2}; \tag{7-27}$$

which implies

$$g_\tau(\tau,\beta) = \frac{2(-1+\beta\tau)}{\beta}, \tag{7-28}$$

and

$$g_\beta(\tau,\beta) = -\frac{2\tau(1-\beta\tau)}{\beta^2} - \frac{2(1-\beta\tau)^2}{\beta^3} = \frac{-2\beta\tau + 2\beta^2\tau^2 - 2(1-\beta\tau)^2}{\beta^3}. \tag{7-29}$$

Additionally,

$$(f_\tau h - f h_\tau) = -2(4 + 4\beta^2\tau^2 - \beta(1+2\tau)^2) \tag{7-30}$$

and

$$(f_\beta h - f h_\beta) = 2\tau(1-2\tau)^2 \tag{7-31}$$

are needed. Substituting Eq. (7-28), Eq. (7-29), Eq. (7-30), and Eq. (7-31) into Eq. (7-4), gives

$$-\frac{4(-1+\beta\tau)^2(-4+\beta+4\beta\tau^2)}{\beta^3} = 0. \tag{7-32}$$

Therefore,

$$\beta = \frac{4}{1+4\tau^2} \tag{7-33}$$

and

$$\alpha = \frac{4\tau}{1+4\tau^2}. \tag{7-34}$$

This gives a relationship beteen α and β,

$$\beta = 2 - 2\sqrt{1-\alpha^2}. \tag{7-35}$$

7.3 TRANSIENT-POSITION-NOISE-REDUCTION COST FUNCTION

To determine which different coefficient relationship are achieved, first consider the transient-position-noise-reduction cost function, J_{NTp}; note that

$$f(\tau,\beta) = 2\beta\tau^2 + (2 - 3\tau\beta), \tag{7-36}$$

which implies

$$f_\tau(\tau,\beta) = 4\beta\tau - 3\beta, \tag{7-37}$$

and

$$f_\beta(\tau,\beta) = 2\tau^2 - 3\tau. \tag{7-38}$$

Also,

$$g(\tau,\beta) = \frac{(2-\beta\tau)(1-\beta\tau)^2}{\tau\beta^2(4-2\beta\tau-\beta)}; \tag{7-39}$$

which implies

$$g_\tau(\tau,\beta) = \frac{2(-4+\beta(1+\tau(4+\beta\tau(3+\beta(-2+\tau(-4+\beta+\beta\tau))))))}{\tau^2\beta^2(4-2\beta\tau-\beta)^2}, \tag{7-40}$$

and

$$g_\beta(\tau,\beta) = \frac{2(-1+\beta\tau)(8+\beta(-3+2\tau(-4+\beta+\beta\tau)))}{\tau\beta^3(4-2\beta\tau-\beta)^2}. \tag{7-41}$$

Additionally,

$$(f_\tau h - f h_\tau) = -2(4+4\beta^2\tau^2-\beta(1+2\tau)^2) \tag{7-42}$$

and

$$(f_\beta h - f h_\beta) = 2\tau(1-2\tau)^2 \tag{7-43}$$

are needed. By substituting Eq. (7-40), Eq. (7-41), Eq. (7-42), and Eq. (7-43) into Eq. (7-4), gives

$$-\frac{4(-1+\beta\tau)^2(-2+\beta\tau(1+\tau))}{\beta^3\tau} = 0. \tag{7-44}$$

Therefore,

$$\beta = \frac{2}{\tau(1+\tau)} \tag{7-45}$$

which is the Benedict-Bordner relationship.

7.4 TRANSIENT-VELOCITY-NOISE-REDUCTION COST FUNCTION

To determine which different coefficient relationship are achieved, first consider the transient–velocity-noise-reduction cost function, J_{NvT}; note that

$$f(\tau,\beta)=\beta, \tag{7-46}$$

which implies

$$f_\tau(\tau,\beta)=0, \tag{7-47}$$

and

$$f_\beta(\tau,\beta)=1. \tag{7-48}$$

Also,

$$g(\tau,\beta)=\frac{-2-2\beta(-1+\tau)\tau+\beta^2\tau^3}{\beta\tau(-4+\beta+2\beta\tau)}; \tag{7-49}$$

which implies

$$\begin{aligned} g_\tau(\tau,\beta) \;&=\; \frac{-2\beta(-1+\tau)-2\beta\tau+3\beta^2\tau^2}{\beta\tau(-4+\beta+2\beta\tau)}-\frac{2(-2-2\beta(-1+\tau)\tau+\beta^2\tau^3)}{\tau(-4+\beta+2\beta\tau)^2} \\ &\quad -\frac{-2-2\beta(-1+\tau)\tau+\beta^2\tau^3}{\beta\tau^2(-4+\beta+2\beta\tau)}, \end{aligned} \tag{7-50}$$

and

$$\begin{aligned} g_\beta(\tau,\beta) \;&=\; \frac{-2(-1+\tau)\tau+2\beta\tau^3}{\beta\tau(-4+\beta+2\beta\tau)}-\frac{(1+2\tau)(-2-2\beta(-1+\tau)\tau+\beta^2\tau^3)}{\beta\tau(-4+\beta+2\beta\tau)^2} \\ &\quad -\frac{-2-2\beta(-1+\tau)+\beta^2\tau^3}{\beta^2\tau(-4+\beta+2\beta\tau)}. \end{aligned} \tag{7-51}$$

Additionally,

$$(f_\tau h-fh_\tau)=\beta(-4+4\tau\beta+\beta) \tag{7-52}$$

and

$$(f_\beta h-fh_\beta)=4\tau \tag{7-53}$$

are needed. By substituting Eq. (7-50), Eq. (7-51), Eq. (7-52), and Eq. (7-53) into Eq. (7-4), gives

$$\frac{4-2\beta\tau(1+\tau))}{\beta\tau}=0. \tag{7-54}$$

Therefore,

$$\beta=\frac{2}{\tau(1+\tau)} \tag{7-55}$$

which is the Benedict-Bordner relationship. Thus, we have the interesting observation that either transient cost function leads to the Benedict-Bordner relationship. It is an interesting intellectual challenge to determine what cost function leads to either the Kalata or to the CTWN relationship.

8 CONCLUSION

The performance of an $\alpha - \beta$ and also an $\alpha - \beta - \gamma$ filter for several deterministic motion models has been analyzed. Closed form solutions have been obtained for the filter performance statistics. The noise reduction ratio is used as a new means of determining filter lag. Cost functionals were used as alternate methods for determining the tracking index. This report presents a general solution to the constant gain tracking filters, which are nothing more than matrix difference equations. The general solution can be used to compute the covariance matrix under very general assumptions about the noise.

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APPENDIX A

EVALUATING SUMS

In this appendix, we evaluate the sums that occur in the main body of this report. The sums are of the form of sums of powers of sines or cosines. We use the complex representation of the trigonometric form in order to simplify these calculations.

Let $y = re^{j\theta}$, then

$$F(y) = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} \tag{A-1}$$

if $-1 \leq \mathrm{Re}(y) \leq 1$. This can be used to evaluate trigonometric sums by noting

$$\mathrm{Im}(F(y)) = S(r,\theta) = \sum_{k=0}^{\infty} r^k \sin(k\theta) \tag{A-2}$$

and

$$\mathrm{Re}(F(y)) = C(r,\theta) = \sum_{k=0}^{\infty} r^k \cos(k\theta) \tag{A-3}$$

The individual sums can be evaluated by applying the definition

$$S(r,\theta) = \mathrm{Im}(F(y)) = \mathrm{Im}\left(\frac{1}{1-y}\right) = \frac{r\sin(\theta)}{1-2r\cos(\theta)+r^2} = \frac{r\sin(\theta)}{\beta} \tag{A-4}$$

and

$$C(r,\theta) = \mathrm{Re}(F(y)) = \mathrm{Re}\left(\frac{1}{1-y}\right) = \frac{1-r\cos(\theta)}{1-2r\cos(\theta)+r^2} = \frac{1-r\cos(\theta)}{\beta} \tag{A-5}$$

where

$$\beta = 1 - 2r\cos(\theta) + r^2. \tag{A-6}$$

For finite sums, define the sum as

$$\begin{aligned} F(y,k) &= \sum_{n=0}^{k} y^n \\ &= \sum_{n=0}^{k} \frac{y^{k+1}-1}{y-1} \end{aligned} \tag{A-7}$$

Hence, we have

$$S(r,\theta) = \mathrm{Im}(F(y)) = \mathrm{Im}\left(\frac{y^{k+1}-1}{y-1}\right) = \frac{r^{k+2}\sin(k\theta) + r\sin(\theta) - r^{k+1}\sin(k+1)\theta}{\beta} \tag{A-8}$$

and

$$C(r,\theta) = \mathrm{Re}(F(y)) = \mathrm{Re}\left(\frac{y^{k+1}-1}{y-1}\right) = \frac{r^{k+2}\cos(k)\theta + r^{k+1}\cos(k+1)\theta + r\cos(\theta) - r^2}{\beta}. \tag{A-9}$$

The first sum can be written as follows:

$$\begin{aligned} S(r,\theta) &= \sum_{m=0}^{k-1} r^m \sin(m\theta) = \mathrm{Im}\left[F(y,k)\right], \qquad \text{(A-10)} \\ &= \mathrm{Im}\left[\frac{(y^k-1)}{(y-1)}\right], \\ &= \mathrm{Im}\left[\frac{r\sin(\theta) + r^{k+1}\sin((k-1)\theta) - r^k\sin(k\theta)}{1-2r\cos(\theta)+r^2}\right] \end{aligned}$$

$$\begin{aligned} C(r,\theta) &= \sum_{m=0}^{k-1} r^m \cos(m\theta) = \mathrm{Re}\left[F(y,k)\right], \qquad \text{(A-11)} \\ &= \mathrm{Re}\left[\frac{(y^k-1)}{(y-1)}\right], \\ &= \mathrm{Re}\left[\frac{1 - r\cos(\theta) + r^{k+1}\cos((k-1)\theta) - r^k\cos(k\theta)}{1-2r\cos(\theta)+r^2}\right]. \end{aligned}$$

We define $S_1(r,\theta)$ as

$$S_1(r,\theta) = \sum_{m=0}^{k-1} m r^m \sin(m\theta), \tag{A-12}$$

which equals

$$\begin{aligned} &= r\frac{\partial}{\partial r}\sum_{m=0}^{k-1} r^m \sin(m\theta) \qquad \text{(A-13)} \\ &= \frac{\alpha r \sin(\theta) + (\alpha+\beta k) r^{k+1} \sin((k-1)\theta) - (\alpha - \beta(1-k)) r^k \sin(k\theta)}{\beta^2}. \end{aligned}$$

Similarly, we define $C_1(r,\theta)$ as

$$C_1(r,\theta) = \sum_{m=0}^{k-1} m r^m \cos(m\theta) \tag{A-14}$$

which equals

$$\begin{aligned} &= r\frac{\partial}{\partial r}\sum_{m=0}^{k-1} r^m \cos(m\theta) \qquad \text{(A-15)} \\ &= \frac{-2r^2 + (2-\alpha) r \cos(\theta) + [\alpha + \beta k]\, r^{k+1} \cos((k-1)\theta) - [\alpha - \beta(1-k)] r^k \cos(k\theta)}{\beta^2}. \end{aligned}$$

We define $S_2(r,\theta)$ as

$$S_2(r,\theta) = \sum_{m=0}^{k-1} m^2 r^m \sin(m\theta) \tag{A-16}$$

which can be evaluated as

$$= m\left(r\frac{\partial}{\partial r}S_1(r,\theta)\right) \tag{A-17}$$

to give

$$= \quad \{(2\alpha^2 + \alpha\beta - 2\beta)r\sin(\theta) \tag{A-18}$$

$$+(2\alpha^2 - 2\beta + \alpha\beta + 2\alpha\beta k + k^2\beta^2)r^{k+1}\sin((k-1)\theta)$$

$$-(2\alpha^2 - 2\beta - \alpha\beta + 2\alpha\beta k + \beta^2 - 2k\beta^2 + k^2\beta^2)r^k\sin(k\theta)\}\}\backslash\beta^3$$

Similarly, we define $C_2(r,\theta)$ as

$$C_2(r,\theta) = \sum_{m=0}^{k-1} m^2 r^m \cos(m\theta) \tag{A-19}$$

which can be evaluated as

$$= m\left(r\frac{\partial}{\partial r}C_1(r,\theta)\right) \tag{A-20}$$

to give

$$= \quad \{-4\alpha r^2 + (4\alpha - 2\alpha^2 - \alpha\beta)r\cos(\theta) \tag{A-21}$$

$$+(2\alpha^2 - 2\beta + \alpha\beta + 2\alpha\beta k + k^2\beta^2)r^{k+1}\cos((k-1)\theta)$$

$$-(2\alpha^2 - 2\beta - \alpha\beta + 2\alpha\beta k + \beta^2 - 2k\beta^2 + k^2\beta^2)r^k\cos(k\theta)\}\backslash\beta^3.$$

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